Méthodes d'Éclatement d'Opérateurs Primales-Duales Monotones au Sens de Bregman

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PREMIÈRE PARTIE : ÉCLATEMENT DANS LES ESPACES HILBERTIENS

Monotone operators

- $\mathcal{H}$ a real Hilbert space.
- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone: for every $(x, x^*) \in \mathcal{H}^2$,

$$(x, x^*) \in \operatorname{gra} A \quad \Leftrightarrow \quad (\forall (y, y^*) \in \operatorname{gra} A) \quad \langle x - y \mid x^* - y^* \rangle \geqslant 0$$

- The resolvent of A, $J_A = (\mathrm{Id} + A)^{-1}\colon \mathcal{H} \to \mathcal{H}$, is firmly nonexpansive and $\operatorname{Fix} J_A = \operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$
- Minty's parametrization:

$$(\forall x \in \mathcal{H}) \quad (J_A x, x - J_A x) = (J_A x, J_{A^{-1}} x) \in \operatorname{gra} A$$

Solving monotone inclusions

- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone,
- Problem: solve: $0 \in Ax$
- Conceptual solution methods when A is simple:
  - The proximal point algorithm (implicit):

$$x_{n+1} = (\mathrm{Id} + \gamma_n A)^{-1} x_n = J_{\gamma_n A} x_n, \quad \text{where} \quad \gamma_n > 0.$$

  - If $A\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive ($A^{-1} - \beta\mathrm{Id}$ is monotone), the explicit iteration

$$x_{n+1} = x_n - \gamma_n A x_n, \quad \text{where} \quad 0 < \gamma_n < 2\beta.$$

- For "real" problems **splitting** is required.

Splitting structured problems: 3 basic methods

$A, B\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, solve $0 \in A\overline{x} + B\overline{x}$.

- Douglas-Rachford splitting (1979)

$$\left\lfloor \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + z_n - y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ β-cocoercive: forward-backward splitting (1979+)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 2\beta \\ y_n = x_n - \gamma_n B x_n \\ x_{n+1} = J_{\gamma_n A} y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ μ-Lipschitzian: forward-backward-forward splitting (2000)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 1/\mu \\ y_n = x_n - \gamma_n B x_n \\ z_n = J_{\gamma_n A} y_n \\ r_n = z_n - \gamma_n B z_n \\ x_{n+1} = x_n - y_n + r_n \end{array} \right.$$

- Spingarn's method (1983) for $0 \in A_1\overline{x} + \cdots + A_n\overline{x}$.

Forward-backward splitting for minimization

- Solution set: $Z = \text{Argmin}\,(f+g)$, where $f \in \Gamma_0(\mathcal{H})$, $g\colon \mathcal{H} \to \mathbb{R}$ convex, differentiable, ∇g is $1/\beta$-Lipschitz-continuous
- The sequence constructed by the algorithm

$$x_{n+1} = \quad \text{prox}_{\gamma \;\; f}\big(x_n - \gamma \quad (\nabla g(x_n) \quad)\big)$$

  - $0 < \gamma < 2\beta$ (Mercier, 1979)

converges weakly to a point in Z

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- The sequence constructed by the algorithm

$$x_{n+1} = \text{prox}_{\gamma_n f}\big(x_n - \gamma_n (\nabla g(x_n) + b_n)\big) + a_n$$

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  - $0 < \inf_{n\in\mathbb{N}} \gamma_n \leqslant \sup_{n\in\mathbb{N}} \gamma_n < 2\beta$ (Tseng, 1990)
  - $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$ (PLC, 2004)

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- The sequence constructed by the algorithm

$$x_{n+1} = x_n + \lambda_n\Big(\text{prox}_{\gamma_n f}\big(x_n - \gamma_n (\nabla g(x_n) + b_n)\big) + a_n - x_n\Big)$$

  - $0 < \gamma < 2\beta$ (Mercier, 1979)
  - $0 < \inf_{n\in\mathbb{N}} \gamma_n \leqslant \sup_{n\in\mathbb{N}} \gamma_n < 2\beta$ (Tseng, 1990)
  - $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$ (PLC, 2004)
  - $(\lambda_n)_{n\in\mathbb{N}}$ in $]0,1]$, $\inf_{n\in\mathbb{N}} \lambda_n > 0$ (PLC, 2004)

converges weakly to a point in Z

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- The sequence constructed by the algorithm

$$x_{n+1} = x_n + \lambda_n\Big(\text{prox}^{U_n}_{\gamma_n f}\big(x_n - \gamma_n U_n^{-1}(\nabla g(x_n) + b_n)\big) + a_n - x_n\Big)$$

  - $0 < \gamma < 2\beta$ (Mercier, 1979)
  - $0 < \inf_{n\in\mathbb{N}} \gamma_n \leqslant \sup_{n\in\mathbb{N}} \gamma_n < 2\beta$ (Tseng, 1990)
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  - $(\lambda_n)_{n\in\mathbb{N}}$ in $]0,1]$, $\inf_{n\in\mathbb{N}} \lambda_n > 0$ (PLC, 2004)
  - $(1+\eta_n)U_{n+1} \succcurlyeq U_n = U_n^* \succcurlyeq \alpha\text{Id}$, $\alpha > 0$, $\eta_n \geqslant 0$, $\sum_{n\in\mathbb{N}} \eta_n < +\infty$ (PLC&Vũ, 2012)

converges weakly to a point in Z

Forward-backward splitting for minimization

- Solution set: $Z = \operatorname{Argmin}(f+g)$, where $f \in \Gamma_0(\mathcal{H})$, $g\colon \mathcal{H} \to \mathbb{R}$ convex, differentiable, ∇g is $1/\beta$-Lipschitz-continuous
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  converges weakly to a point in Z
- Also: almost surely weakly convergent versions with random block-coordinate sweeping (PLC&Pesquet, SIOPT, July 2015) and/or stochastic approximations (PLC&Pesquet, Pure Appl. Funct. Anal., Jan. 2016)

Further properties of forward-backward splitting

- Solution set: $Z = $Argmin $f + g$
- $x_{n+1} = x_n + \lambda_n\Big(\text{prox}_{\gamma_n f}(x_n - \gamma_n \nabla g(x_n)) - x_n\Big)$, $\varepsilon \leqslant \gamma_n \leqslant (2-\varepsilon)\beta$
- $(\forall n \in \mathbb{N})(\forall z \in Z) \quad \|x_{n+1} - z\| \leqslant \|x_n - z\|$: Fejér monotonicity
- $(\forall n \in \mathbb{N}) \quad (f+g)(x_{n+1}) \leqslant (f+g)(x_n)$
- Convergence is only weak
- Even in the finite dimensional or the linear case, no (upper bound on the worst) rate of convergence of $\|x_n - x_\infty\|$ exists
- $\sum_{n\in\mathbb{N}} \big|(f+g)(x_n) - \inf(f+g)(\mathcal{H})\big|^2 < +\infty$
- If $\sum_{n\in\mathbb{N}}(1-\lambda_n) < +\infty$, $(f+g)(x_n) - \inf(f+g)(\mathcal{H}) = o(1/n)$ (PLC, Salzo, Villa, 2017)
- In the case of the projected gradient method, some form of the above results already in:
  - E. S. Levitin and B. T. Polyak, Constrained minimization methods, *Comput. Math. Math. Phys.*, vol. 6, pp. 1–50, 1966

On minimizing sequences

- Let $\Phi \in \Gamma_0(\mathcal{H})$, $Z = \operatorname{Argmin} \Phi \neq \varnothing$ the solution set
- Minimizing sequences may have little to do with actually approaching a point in Z as we can have (even in $\mathbb{R}^2$):
  - $\Phi(x_n) \to \inf \Phi(\mathcal{H})$ and $(\forall z \in Z)\ \|x_n - z\| \geqslant 1$
  - $\Phi(x_n) \to \inf \Phi(\mathcal{H})$ and $(\forall z \in Z)\ \|x_n - z\| \to +\infty$
  - ... and vice versa $\Phi(x_n) \equiv +\infty$ and $x_n \to z \in Z$
- The whole area of metric regularity addresses such issues

Splitting algorithms (1979-2000)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + B\overline{x}$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone

Splitting algorithms (Briceño-Arias-PLC, 2011)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + L^* B(L\overline{x} - r)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ is maximally monotone, $r \in \mathcal{G}$, $L \in \mathscr{B}(\mathcal{H}, \mathcal{G})$

Splitting algorithms (Briceño-Arias-PLC, 2011)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^* B_k (L_k \overline{x} - r_k)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k : \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$

Splitting algorithms (PLC-Pesquet, 2012)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^*(B_k \,\square\, D_k)(L_k\overline{x} - r_k)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$
- $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$

Splitting algorithms (PLC-Pesquet, 2012)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^*(B_k \,\square\, D_k)(L_k - r_k\overline{x}) + C\overline{x}$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$
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- $C\colon \mathcal{H} \to \mathcal{H}$ is monotone and μ-Lipschtizian

Splitting algorithms (PLC, 2013)

find $\overline{x}_1 \in \mathcal{H}_1, \ldots, \overline{x}_m \in \mathcal{H}_m$ such that

$$\begin{cases} z_1^* \in A_1\overline{x}_1 + \sum_{k=1}^{K} L_{k1}^*\Big((B_k \,\Box\, D_k)\Big(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Big)\Big) + C_1\overline{x}_1 \\ \quad\vdots \\ z_m^* \in A_m\overline{x}_m + \sum_{k=1}^{K} L_{km}^*\Big((B_k \,\Box\, D_k)\Big(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Big)\Big) + C_m\overline{x}_m \end{cases}$$

where:

- $z_i^* \in \mathcal{H}_i$, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ is maximally monotone
- $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$
- $D_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\Box\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$
- $C_i \colon \mathcal{H}_i \to \mathcal{H}_i$ is monotone and μ_i-Lipschtizian

Splitting algorithms (PLC, 2013)

- $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_p$
- $\boldsymbol{M} \colon \mathcal{K} \to 2^{\mathcal{K}} \colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto (-z_1^* + A_1 x_1) \times \cdots \times (-z_m^* + A_m x_m) \times (r_1 + B_1^{-1} v_1^*) \times \cdots \times (r_p + B_p^{-1} v_p^*)$
- $\boldsymbol{Q} \colon \mathcal{K} \to \mathcal{K} \colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto \big(C_1 x_1 + \sum_{k=1}^K L_{k1}^* v_k^*, \ldots, C_m x_m + \sum_{k=1}^K L_{km}^* v_k^*, -\sum_{i=1}^m L_{1i} x_i + D_1^{-1} v_1^*, \ldots, \sum_{i=1}^m L_{Ki} x_i + D_K^{-1} v_K^*\big)$
- $\boldsymbol{M}$ and $\boldsymbol{Q}$ are maximally monotone, $\boldsymbol{Q}$ is Lipschitzian, the zeros of $\boldsymbol{M} + \boldsymbol{Q}$ are primal-dual solutions
- Solve $\boldsymbol{0} \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{x}$, where $\boldsymbol{x} = (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*)$ via Tseng's forward-backward-forward splitting algorithm

$$
\left\lfloor
\begin{array}{l}
\boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{Q}\boldsymbol{x}_n \\
\boldsymbol{p}_n = (\mathrm{Id} + \boldsymbol{M})^{-1} \boldsymbol{y}_n \\
\boldsymbol{q}_n = \boldsymbol{p}_n - \boldsymbol{Q}\boldsymbol{p}_n \\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n
\end{array}
\right.
$$

in $\mathcal{K}$ to get...

Splitting algorithms (PLC, 2013)

For $n = 0, 1, \ldots$

$$\varepsilon \leqslant \gamma_n \leqslant (1-\varepsilon)/\Big(\max\Big\{\max_{1\leqslant i\leqslant m} \mu_i, \max_{1\leqslant k\leqslant K} \nu_k\Big\} + \sqrt{\textstyle\sum_{k=1}^K \sum_{i=1}^m \|L_{ki}\|^2}\Big)$$

For $i = 1, \ldots, m$

$$s_{1,i,n} = x_{i,n} - \gamma_n\Big(C_i x_{i,n} + \textstyle\sum_{k=1}^K L_{ki}^* v_{k,n}^*\Big)$$

$$p_{1,i,n} = J_{\gamma_n A_i}(s_{1,i,n} + \gamma_n z_i)$$

For $k = 1, \ldots, K$

$$s_{2,k,n} = v_{k,n}^* - \gamma_n\Big(D_k^{-1} v_{k,n}^* - \textstyle\sum_{i=1}^m L_{ki} x_{i,n}\Big)$$

$$p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + J_{\gamma_n^{-1} B_k}(\gamma_n^{-1} s_{2,k,n} - r_k)\big)$$

$$q_{2,k,n} = p_{2,k,n} - \gamma_n\Big(D_k^{-1} p_{2,k,n} - \textstyle\sum_{i=1}^m L_{ki} p_{1,i,n}\Big)$$

$$v_{k,n+1}^* = v_{k,n}^* - s_{2,k,n} + q_{2,k,n}$$

For $i = 1, \ldots, m$

$$q_{1,i,n} = p_{1,i,n} - \gamma_n\Big(C_i p_{1,i,n} + \textstyle\sum_{k=1}^K L_{ki}^* p_{2,k,n}\Big)$$

$$x_{i,n+1} = x_{i,n} - s_{1,i,n} + q_{1,i,n}$$

Some limitations of the state-of-the-art

We present a new framework that circumvents simultaneously the limitations of current methods, which require:

- inversions of linear operators or knowledge of bounds on norms of all the L_{ki}
- the proximal parameters must be the same for all the monotone operators
- activation of the resolvents of all the monotone operators: impossible in huge-scale problems
- synchronicity: all resolvent operator evaluations must be computed and used during the current iteration

and, in general,

- converge only weakly

Asynchronous, block-iterative splitting

- For every $i \in I$ (finite), $\mathcal{H}_i$ a Hilbert space, $A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ maximally monotone, $z_i^* \in \mathcal{H}_i$
- For every $k \in K$ (finite), $\mathcal{G}_k$ a Hilbert space, $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ maximally monotone, $r_k \in \mathcal{G}_k$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$
- **Initial problem:** find $(\overline{x}_i)_{i\in I} \in \boldsymbol{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}_i$ such that

$$(\forall i \in I) \quad z_i^* \in A_i\overline{x}_i + \sum_{k\in K} L_{ki}^*\Bigg(B_k\Bigg(\sum_{j\in I} L_{kj}\overline{x}_j - r_k\Bigg)\Bigg)$$

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- **Initial problem:** find $(\overline{x}_i)_{i \in I} \in \boldsymbol{\mathcal{H}} = \bigoplus_{i \in I} \mathcal{H}_i$ such that

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- **Dual problem:** find $(\overline{v}_k^*)_{k \in K} \in \boldsymbol{\mathcal{G}} = \bigoplus_{k \in K} \mathcal{G}_k$ such that

$$(\forall k \in K) \quad -r_k \in -\sum_{i \in I} L_{ki} \Bigg(A_i^{-1} \Bigg(z_i^* - \sum_{l \in K} L_{li}^* \overline{v}_l^* \Bigg) \Bigg) + B_k^{-1} \overline{v}_k^*$$

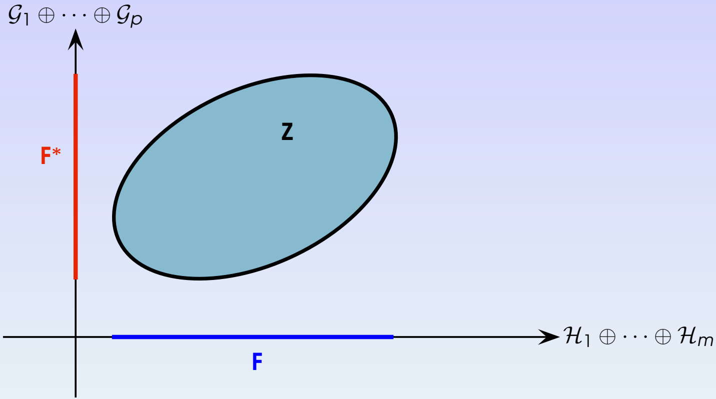
Asynchronous, block-iterative splitting

- **Solutions set:** the associated Kuhn-Tucker set

$$\boldsymbol{Z} = \Big\{ \big((\overline{x}_i)_{i\in I}, (\overline{v}_k^*)_{k\in K}\big) \;\Big|\; \overline{x}_i \in \mathcal{H}_i \;\text{ and }\; z_i^* - \sum_{k\in K} L_{ki}^* \overline{v}_k^* \in A_i \overline{x}_i, \quad \overline{v}_k^* \in \mathcal{G}_k \;\text{ and }\; \sum_{i\in I} L_{ki}\overline{x}_i - r_k \in B_k^{-1}\overline{v}_k^* \Big\}$$

- $\boldsymbol{Z}$ is a closed convex set
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{H}}$ is the set $\boldsymbol{F}$ of primal solutions
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{G}}$ is the set $\boldsymbol{F}^*$ of dual solutions

The Kuhn-Tucker set



With proper CQ, this framework includes..

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{(x_i)_{i\in I}\in\boldsymbol{\mathcal{H}}}{\text{minimize}} \sum_{i\in I}\big(f_i(x_i)-\langle x_i \mid z_i^*\rangle\big)+\sum_{k\in K} g_k\Big(\sum_{i\in I} L_{ki}x_i - r_k\Big)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$

- Let $\boldsymbol{F}^*$ be the set of solutions to the dual problem

$$\underset{(v_k^*)_{k\in K}\in\bigoplus_{k\in K}\mathcal{G}_k}{\text{minimize}} \sum_{i\in I} f_i^*\Big(z_i^* - \sum_{k\in K} L_{ki}^* v_k^*\Big)+\sum_{k\in K}\big(g_k^*(v_k^*)+\langle v_k^* \mid r_k\rangle\big)$$

- Associated Kuhn-Tucker set

$$\boldsymbol{Z}=\Big\{\big((\overline{x}_i)_{i\in I},(\overline{v}_k^*)_{k\in K}\big)\;\Big|\;\overline{x}_i\in\mathcal{H}_i \text{ and } z_i^*-\sum_{k\in K}L_{ki}^*\overline{v}_k^*\in\partial f_i(\overline{x}_i),\quad \overline{v}_k^*\in\mathcal{G}_k \text{ and } \sum_{i\in I}L_{ki}\overline{x}_i - r_k \in \partial g_k^*(\overline{v}_k^*)\Big\}$$

Methodology for 2 operators

- $0 \in A\overline{x} + L^*\big(BL\overline{x}\big)$ and $0 \in -L\big(A^{-1}\big(-L^*\overline{v}^*\big)\big) + B^{-1}\overline{v}^*$

Methodology for 2 operators

- $0 \in A\overline{x} + L^*\big(BL\overline{x}\big)$ and $0 \in -L\big(A^{-1}\big(-L^*\overline{v}^*\big)\big) + B^{-1}\overline{v}^*$
- Take $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$. Then $-L^*\overline{v}^* \in A\overline{x}$ and $L\overline{x} \in B^{-1}\overline{v}^*$, i.e.,

$$\big(\overline{x}, -L^*\overline{v}^*\big) \in \operatorname{gra} A \quad \text{and} \quad \big(L\overline{x}, \overline{v}^*\big) \in \operatorname{gra} B$$

Methodology for 2 operators

- $0 \in A\overline{x} + L^*(BL\overline{x})$ and $0 \in -L(A^{-1}(-L^*\overline{v}^*)) + B^{-1}\overline{v}^*$
- Take $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$. Then $-L^*\overline{v}^* \in A\overline{x}$ and $L\overline{x} \in B^{-1}\overline{v}^*$, i.e.,

$$(\overline{x}, -L^*\overline{v}^*) \in \operatorname{gra} A \quad \text{and} \quad (L\overline{x}, \overline{v}^*) \in \operatorname{gra} B$$

- Suppose that at iteration n

$$(a_n, a_n^*) \in \operatorname{gra} A \quad \text{and} \quad (b_n, b_n^*) \in \operatorname{gra} B$$

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$$\langle a_n - \overline{x} \mid a_n^* + L^*\overline{v}^* \rangle + \langle b_n - L\overline{x} \mid b_n^* - \overline{v}^* \rangle \geqslant 0$$

i.e.,

$$\langle (\overline{x}, \overline{v}) \mid (a_n^* + L^* b_n^*, b_n - La_n) \rangle \leqslant \langle a_n \mid a_n^* \rangle + \langle b_n \mid b_n^* \rangle$$

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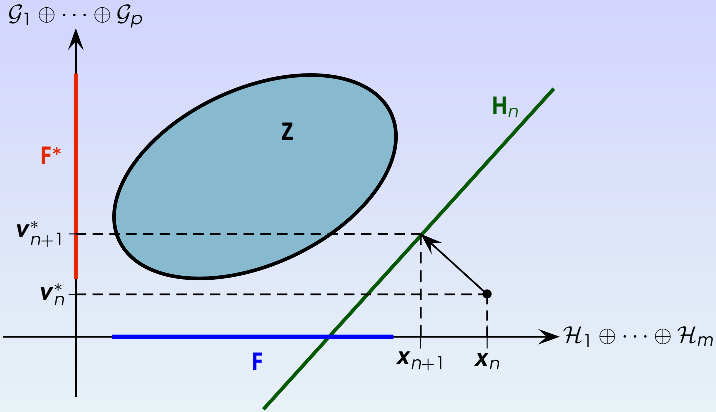
- This places $\boldsymbol{Z}$ in the closed affine half-space with outer normal $(a_n^* + L^*b_n^*, b_n - La_n)$!

Fejér monotone scheme in the general case



- Choose suitable points in the graphs of $(A_i)_{i\in I}$ and $(B_k)_{k\in K}$ to construct a half-space H_n containing Z
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}^*_{n+1}) = P_{\mathsf{H}_n}(\boldsymbol{x}_n, \boldsymbol{v}^*_n) \rightharpoonup (\boldsymbol{x}, \boldsymbol{v}^*) \in \mathsf{Z} \subset \mathsf{F} \times \mathsf{F}^*$

Main novelties

- **Block iterations:** At iteration n, we require calculation of new points in the graphs of only some the operators $(A_i)_{i \in I_n \subset I}$ and $(B_k)_{k \in K_n \subset K}$. The control sequences $(I_n)_{n \in \mathbb{N}}$ and $(K_n)_{n \in \mathbb{N}}$ dictate how frequently the various operators are used.
- **Asynchronicity:** A new point $(a_{i,n}, a^*_{i,n}) \in \operatorname{gra} A_i$ being incorporated into the calculations at iteration n may be based on data $x_{i,c_i(n)}$ and $(v^*_{k,c_i(n)})_{k \in K}$ available at some possibly earlier iteration $c_i(n) \leqslant n$. Therefore, the calculation of $(a_{i,n}, a^*_{i,n})$ could have been initiated at iteration $c_i(n)$, with its results becoming available only at iteration n. Likewise, for $(b_{k,n}, b^*_{k,n}) \in \operatorname{gra} B_k$.

Also:

- No knowledge of the $\|L_{ki}\|$s is required
- No linear operator inversion is required
- No bounds required on the proximal parameters

Asynchronous block-iterative proximal splitting I

for $n = 0, 1, \ldots$

for every $i \in I_n$

$l_{i,n}^* = \sum_{k\in K} L_{ki}^* v_{k,c_i(n)}^*$

$(a_{i,n}, a_{i,n}^*) = \Big(J_{\gamma_{i,c_i(n)} A_i}\big(x_{i,c_i(n)} + \gamma_{i,c_i(n)}(z_i - l_{i,n}^*)\big), \gamma_{i,c_i(n)}^{-1}(x_{i,c_i(n)} - a_{i,n}) - l_{i,n}^*\Big)$

for every $i \in I \smallsetminus I_n$

$(a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*)$

for every $k \in K_n$

$l_{k,n} = \sum_{i\in I} L_{ki} x_{i,d_k(n)}$

$(b_{k,n}, b_{k,n}^*) = \Big(r_k + J_{\mu_{k,d_k(n)} B_k}\big(l_{k,n} + \mu_{k,d_k(n)} v_{k,d_k(n)}^* - r_k\big), v_{k,d_k(n)}^* + \mu_{k,d_k(n)}^{-1}(l_{k,n} - b_{k,n})\Big)$

for every $k \in K \smallsetminus K_n$

$(b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*)$

$\big((t_{i,n}^*)_{i\in I}, (t_{k,n})_{k\in K}\big) = \big((a_{i,n}^* + \sum_{k\in K} L_{ki}^* b_{k,n}^*)_{i\in I}, (b_{k,n} - \sum_{i\in I} L_{ki} a_{i,n})_{k\in K}\big)$

$\tau_n = \sum_{i\in I} \|t_{i,n}^*\|^2 + \sum_{k\in K} \|t_{k,n}\|^2$

if $\tau_n > 0$

$$\theta_n = \frac{\lambda_n}{\tau_n} \max\Big\{0, \sum_{i\in I} \big(\langle x_{i,n} \mid t_{i,n}^*\rangle - \langle a_{i,n} \mid a_{i,n}^*\rangle\big) + \sum_{k\in K} \big(\langle t_{k,n} \mid v_{k,n}^*\rangle - \langle b_{k,n} \mid b_{k,n}^*\rangle\big)\Big\}$$

else $\theta_n = 0$

for every $i \in I$

$x_{i,n+1} = x_{i,n} - \theta_n t_{i,n}^*$

for every $k \in K$

$v_{k,n+1}^* = v_{k,n}^* - \theta_n t_{k,n}$

Convergence

- $(I_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of I, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of K such that $I_0 = I$, $K_0 = K$, and

$$(\forall n \in \mathbb{N}) \quad \left(\bigcup_{j=n}^{n+M-1} I_j = I \quad \text{and} \quad \bigcup_{j=n}^{n+M-1} K_j = K \right). \tag{1}$$

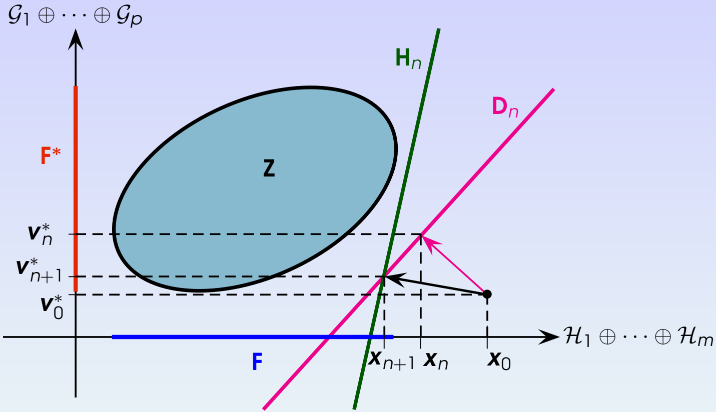
- $(c_i(n))_{n\in\mathbb{N}}$ and $(d_k(n))_{n\in\mathbb{N}}$ are sequences in $\mathbb{N}$ such that

$$(\forall i \in I) \;\; n - D \leqslant c_i(n) \leqslant n \quad \text{and} \quad (\forall k \in K) \;\; n - D \leqslant d_k(n) \leqslant n$$

- $\varepsilon \in \left]0, 1\right[$ and $(\gamma_{i,n})_{n\in\mathbb{N}}$ and $(\mu_{k,n})_{n\in\mathbb{N}}$ are sequences in $[\varepsilon, 1/\varepsilon]$.

Set $x_n = (x_{i,n})_{i\in I}$ and $v_n^* = (v_{k,n}^*)_{k\in K}$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{x} \in \boldsymbol{F}$, $(v_n^*)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{v} \in \boldsymbol{F}^*$, and $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$.

Asynchronous block-iterative proximal splitting II



- Construct $\mathbf{H}_n$ as before
- The half-space $\mathbf{D}_n$ satisfies $(\boldsymbol{x}_n, \boldsymbol{v}_n^*) = P_{\mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*)$
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}_{n+1}^*) = P_{\mathbf{H}_n \cap \mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \to P_{\mathbf{Z}}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \in \mathbf{F} \times \mathbf{F}^*$

DEUXIÈME PARTIE : ÉCLATEMENT DANS LES ESPACES DE BANACH

Legendre functions (Bauschke-Borwein-PLC, 2001)

- $\mathcal{X}$ is a reflexive real Banach space.
- $f \in \Gamma_0(\mathcal{X})$ is:
  - essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
  - essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$;
  - a Legendre function, if it is both essentially smooth and essentially strictly convex.

Legendre functions

If f is a Legendre function, then

- f^* is a Legendre function.
- $\operatorname{dom} \partial f = \operatorname{int} \operatorname{dom} f \neq \varnothing$ and f is Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f$.
- $\nabla f \colon \operatorname{int} \operatorname{dom} f \to \operatorname{int} \operatorname{dom} f^*$ is bijective with inverse

$$\nabla f^* \colon \operatorname{int} \operatorname{dom} f^* \to \operatorname{int} \operatorname{dom} f.$$

- For every sequence $(x_n)_{n \in \mathbb{N}}$ in $\operatorname{int} \operatorname{dom} f$ that converges to some point in $\operatorname{bdry} \operatorname{dom} f$, we have

$$\|\nabla f(x_n)\| \to +\infty,$$

Legendre functions

- Let $f\colon \mathcal{X} \to \left]-\infty,+\infty\right]$ be a Legendre function.
- The associated Bregman distance is

$$D^f\colon \mathcal{X}\times\mathcal{X} \to [0,+\infty]$$
$$(x,y) \mapsto \begin{cases} f(x)-f(y)-\langle x-y, \nabla f(y)\rangle, & \text{if } \; y\in \operatorname{int}\operatorname{dom} f;\\ +\infty, & \text{otherwise.}\end{cases}$$

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ is Bregman-monotone with respect to $Z\subset\mathcal{X}$ if (Bauschke-Borwein-PLC, 2003)
  - $Z\cap\operatorname{dom} f\neq\varnothing$.
  - $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{int}\operatorname{dom} f$.
  - $(\forall x\in Z\cap\operatorname{dom} f)(\forall n\in\mathbb{N})\;\; D(x,x_{n+1})\leqslant D(x,x_n)$.

How to generate a Bregman monotone sequence?

- For every x and y in $\operatorname{int}\operatorname{dom} f$, set

$$H(x,y) = \big\{z \in \mathcal{X} \mid \langle z - y, \nabla f(x) - \nabla f(y)\rangle \leqslant 0\big\}.$$

- Call $\mathfrak{B}$ the class of all operators $T\colon \mathcal{X} \to 2^{\mathcal{X}}$ such that

$$\operatorname{ran} T \subset \operatorname{dom} T = \operatorname{intdom} f, \text{ and } (\forall (x,u) \in \operatorname{gra} T)\ \operatorname{Fix} T \subset H(x,u).$$

- Set

$$x_0 \in \operatorname{int}\operatorname{dom} f \quad \text{and} \quad (\forall n \in \mathbb{N})\ \ x_{n+1} \in T_n x_n, \quad \text{where} \quad T_n \in \mathfrak{B}.$$

- Then $(x_n)_{n\in\mathbb{N}}$ is Bregman monotone with respect to

$$Z = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n.$$

Bregman Projection

- Let $f\colon \mathcal{X} \to \,]-\infty, +\infty]$ be a Legendre function.
- Let C be a closed convex subset of $\mathcal{X}$ such that $C \cap \operatorname{int}\operatorname{dom} \neq \varnothing$.
- The Bregman projector onto C induced by f is

$$
\begin{aligned}
P_C^f \colon \operatorname{int}\operatorname{dom} f &\to C \cap \operatorname{int}\operatorname{dom} f \\
y &\mapsto \underset{x \in C}{\operatorname{argmin}}\, D^f(x, y).
\end{aligned}
$$

- $P_C^f \in \mathfrak{B}$.

Bregman Resolvent

- Let $f\colon \mathcal{X} \to]-\infty, +\infty]$ be a Legendre function.
- Let $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ be maximally monotone.
- The Bregman resolvent of A induced by f is

$$R_A^f = (\nabla f + A)^{-1} \circ \nabla f\colon \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f.$$

- $R_A^f \in \mathfrak{B}$.
- The Bregman proximal operator of $\varphi \in \Gamma_0(\mathcal{X})$ induced by f is

$$\operatorname{prox}_\varphi^f = R_{\partial\varphi}^f\colon \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$$
$$y \mapsto \underset{x\in\mathcal{X}}{\operatorname{argmin}}\, \varphi(x) + D^f(x,y).$$

Note: Even in Euclidean spaces, it may be easier to evaluate $\operatorname{prox}_\varphi^f$ than the usual Moreau proximity operator $\operatorname{prox}_\varphi = (\operatorname{Id} + \gamma\partial\varphi)^{-1}$. Here are a few examples...

Examples of Bregman proximity operators (PLC-Nguyen, 2016)

Suppose that f is the Fermi-Dirac entropy function on $\mathbb{R}^m$.

- Let $\omega \in \mathbb{R}$ and suppose that each component of φ is

$$\phi\colon \xi \mapsto \begin{cases} \xi \ln \xi - \omega\xi, & \text{if } \xi \in \left]0,+\infty\right[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

  Then $\eta_i = -e^{\xi_i+\omega-1}/2 + \sqrt{e^{2(\xi_i+\omega-1)}/4 + e^{\xi_i+\omega-1}}$.

- Suppose that

$$\phi\colon \xi \mapsto \begin{cases} (1-\xi)\ln(1-\xi) + \xi, & \text{if } \xi \in \left]-\infty,1\right[; \\ 1, & \text{if } \xi = 1; \\ +\infty, & \text{otherwise.} \end{cases}$$

  Then $\eta_i = 1 + e^{-\xi_i}/2 - \sqrt{e^{-\xi_i} + e^{-2\xi_i}/4}$.

Problem

- Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexive real Banach spaces, let $\boldsymbol{\mathcal{X}} = \mathcal{X} \times \mathcal{Y}^*$ be normed with $(x, y^*) \mapsto \sqrt{\|x\|^2 + \|y^*\|^2}$, and let $\boldsymbol{\mathcal{X}}^*$ be its topological dual, that is, $\mathcal{X}^* \times \mathcal{Y}$ equipped with the norm $(x^*, y) \mapsto \sqrt{\|x^*\|^2 + \|y\|^2}$.
- Let $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ and $B\colon \mathcal{Y} \to 2^{\mathcal{Y}^*}$ be maximally monotone, and let $L\colon \mathcal{X} \to \mathcal{Y}$ be linear and bounded.
- Primal problem:

$$\text{find} \quad x \in \mathcal{X} \quad \text{such that} \quad 0 \in Ax + L^*BLx.$$

- Dual problem:

$$\text{find} \quad y^* \in \mathcal{Y}^* \quad \text{such that} \quad 0 \in -LA^{-1}(-L^*y^*) + B^{-1}y^*.$$

- Kuhn-Tucker set:

$$\boldsymbol{Z} = \big\{(x, y^*) \in \boldsymbol{\mathcal{X}} \mid -L^*y^* \in Ax \quad \text{and} \quad Lx \in B^{-1}y^*\big\}.$$

Problem

- Let $f\colon \mathcal{X} \to]-\infty,+\infty]$ and $g\colon \mathcal{Y} \to]-\infty,+\infty]$ be Legendre functions, set

$$\boldsymbol{f}\colon \mathcal{X} \to]-\infty,+\infty]\colon (x,y^*) \mapsto f(x)+g^*(y^*),$$

  let $x_0 \in \operatorname{int}\operatorname{dom} f$, let $y_0^* \in \operatorname{int}\operatorname{dom} g^*$.
- Suppose that $\boldsymbol{Z} \cap \operatorname{int}\operatorname{dom} \boldsymbol{f} \neq \varnothing$.
- The problem is to find the best Bregman approximation $(\overline{x},\overline{y}^*) = P^{\boldsymbol{f}}_{\boldsymbol{Z}}(x_0,y_0^*)$ to (x_0,y_0^*) from $\boldsymbol{Z}$.

Methodology

- Use the same geometrical intuition as in the hilbertian case.
- Thus, in the fully parallel case we iterate

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
(\gamma_n, \mu_n) \in [\varepsilon, \sigma] \times [\delta, \sigma] \\
a_n = R^h_{\gamma_n A}\big(\nabla h(x_n) - \gamma_n L^* y_n^*\big) \\
a_n^* = \gamma_n^{-1}\big(\nabla h(x_n) - \nabla h(a_n)\big) - L^* y_n^* \\
b_n = R^j_{\mu_n B}\big(\nabla j(L x_n) + \mu_n y_n^*\big) \\
b_n^* = \mu_n^{-1}\big(\nabla j(L x_n) - \nabla j(b_n)\big) + y_n^* \\
\boldsymbol{H}_n = \Big\{(x, y^*) \in \mathfrak{X} \,\Big|\, \langle x, a_n^* + L^* b_n^* \rangle + \langle b_n - L a_n, y^* \rangle \\
\qquad\qquad\qquad\qquad\qquad \leqslant \langle a_n, a_n^* \rangle + \langle b_n, b_n^* \rangle \Big\} \\
(x_{n+1/2}, y_{n+1/2}^*) = P^{\boldsymbol{f}}_{\boldsymbol{H}_n}(x_n, y_n^*) \\
(x_{n+1}, y_{n+1}^*) = Q^{\boldsymbol{f}}\big((x_0, y_0^*), (x_n, y_n^*), (x_{n+1/2}, y_{n+1/2}^*)\big).
\end{array}
\right.
\end{array}
$$

where

$$
Q^{\boldsymbol{f}}(\boldsymbol{x}_0, \boldsymbol{x}, \boldsymbol{y}) = P^{\boldsymbol{f}}_{H^{\boldsymbol{f}}(\boldsymbol{x}_0, \boldsymbol{x}) \cap H^{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{y})} \boldsymbol{x}_0.
$$

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