On the geometry of branchiopod eggs

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PHD thesis under the supervision of Antoine Henrot (IECL Nancy), and Yannick Privat (LJLL)

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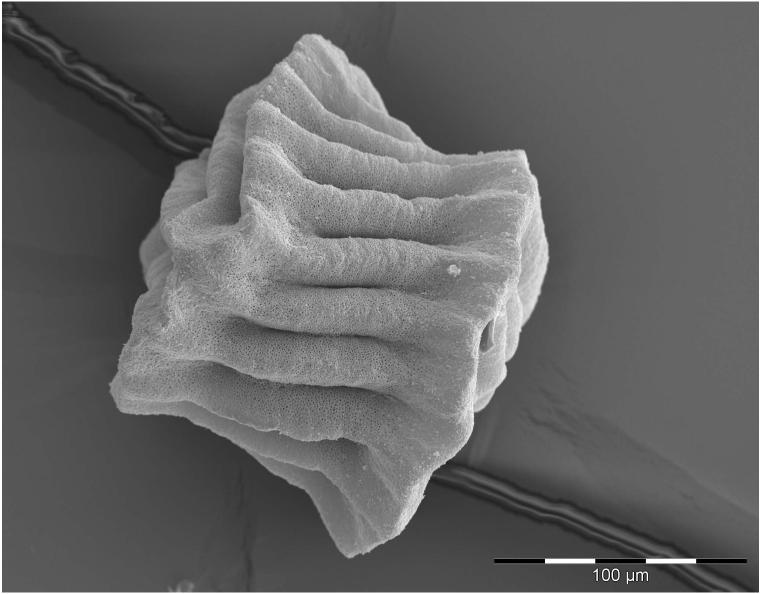
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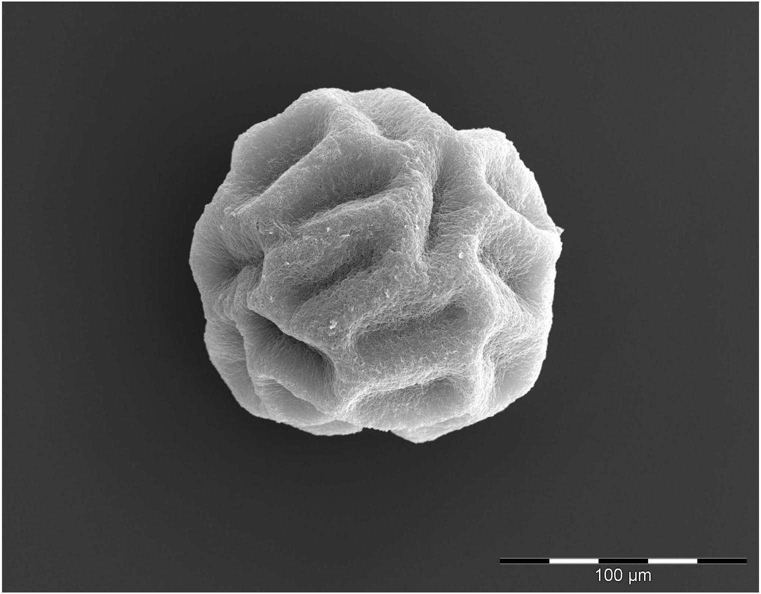
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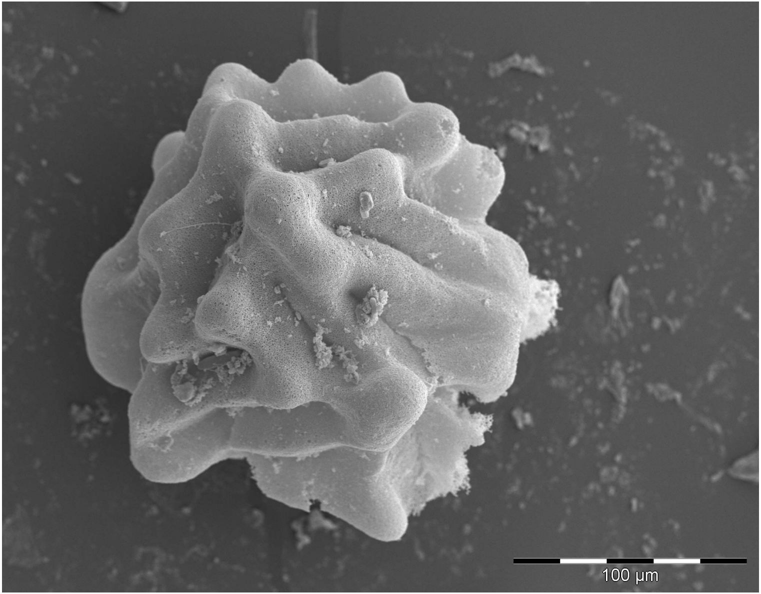
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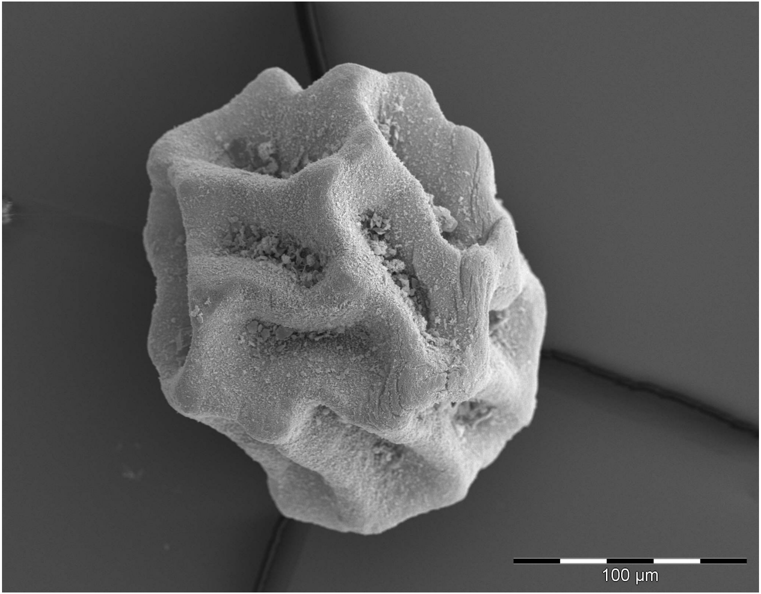
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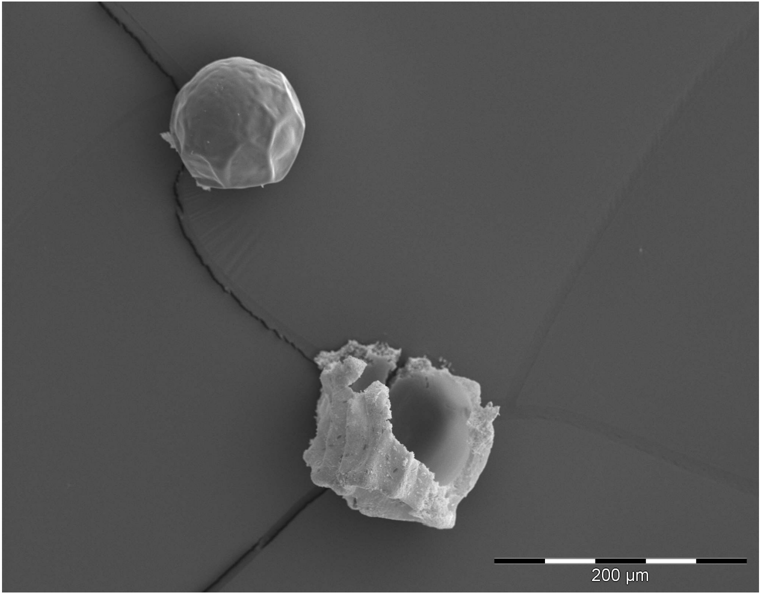
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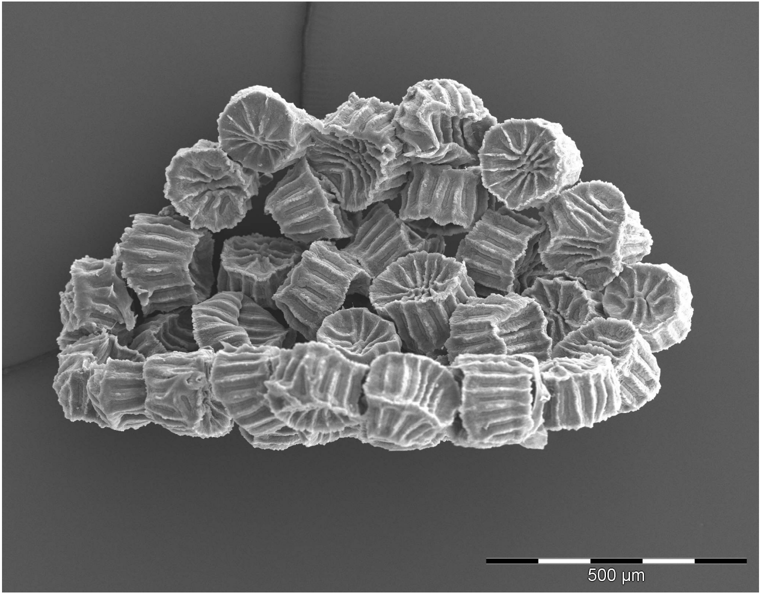

500 µm

Goal : understand the shape of the eggs by using a shape optimization approach

FIGURE: Honeycomb structure as a result of a shape optimization process

Plan

- Propose a realistic explanation of the shape of the eggs.
- Mathematic formulation of the problem : minimization of a geometric functional.
- Solve shape optimization problem.
- Compare the obtained solutions with the actual shape of the eggs.

Problem

Among the different aspects, we try to highlight that the shape of the egg is designed in such a way that the shrimp can incubate the largest number of eggs

This leads to packing problems, i-e find the best way to arrange a collection of objects, and find those who are the ”easiest” to arrange

Packing

From now on we are working on $\mathbb{R}^2$. Let us denote by $\mathcal{K}$ the set of convex bodies of the plan. If $K \in \mathcal{K}$ is a convex body, a packing of K is a set of the form :

$$P(K) = \bigcup_{i \in I} \tau_i(K).$$

Where I is a finite or countable set, and τ_i are direct affine isometries of $\mathbb{R}^2$ such that for all $i \neq j$, $\text{int}(\tau_i(K)) \cap \text{int}(\tau_j(K)) = \emptyset$. K is called the pattern of the packing. We denote by $\mathcal{P}(K)$ the set of packings of K. If $\mathbb{R}^2 \in \mathcal{P}(K)$, then K is a tiling domain.

Density

We want to give a sense to the "density" of a packing, i-e the portion of the space the packing takes.
The usual definition is the following.

$$d(P(K)) = \liminf_{r \to \infty} \frac{\sharp\{i, \tau_i(K) \subset rI^2\}|K|}{r^2}$$

$$\delta_1(K) = \sup_{P(K) \in \mathcal{P}(K)} d(P(K))$$

But this definition is not an easy one to deal with, so we will consider a slightly easier one :

$$\delta(K) = \frac{|K|}{|K^T|}$$

where K^T is the smallest tiling domain circumscribed about K.

Tiling domains

Theorem

The convex tiling domain are in the plane :

- *Triangles*
- *Quadrangles.*
- *Fifteen kinds of pentagons.*
- *Three kinds of hexagons*

Pentagonal tilings

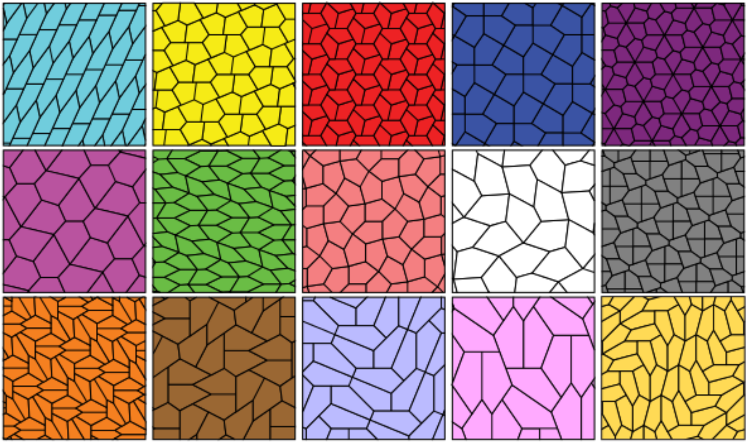


FIGURE: The fifteen kinds of pentagonal tilings

hexagonal tiling

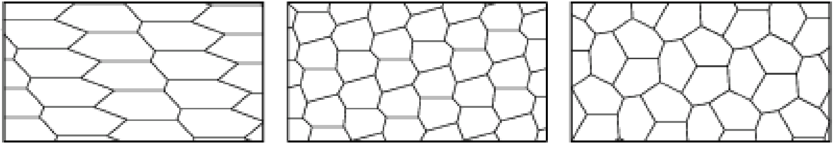


FIGURE: The three kinds of hexagonal tiling

First result on the density

Theorem (Kuperberg G,W [2])

for all $K \in \mathcal{K}$, we have

$$\delta(K) \geq \frac{2}{\sqrt{3}}.$$

Diameter

We also want to give a notion of a non-dispersal property of a packing. We introduce :

$$D^1_\infty(K) = \inf_{P \in \mathcal{P}(K)} \limsup_{R \to \infty} \frac{2R}{\sqrt{\sharp\{i, \tau_i(K) \subset D(0,R)\}} D(K)}.$$

where $D(K)$ is the diameter of K.

Diameter

Theorem

Let $K \in \mathcal{K}$, then

$$\frac{2\sqrt{|K|}}{\sqrt{\pi}D(K)} \leq D^1_\infty(K) \leq \sqrt{\frac{2}{\sqrt{3}}\frac{2\sqrt{|K|}}{\sqrt{\pi}D(K)}}.$$

furthermore, if K is a tiling domain

$$D^1_\infty(K) = \frac{2\sqrt{|K|}}{\sqrt{\pi}D(K)}.$$

Thus, we we chose as a more tractable criterion :

$$D_\infty(K) = \frac{2\sqrt{|K|}}{\sqrt{\pi}Diam(K)}.$$

Inradius

The shell of the eggs is supposed to protect the membrane of the embryo (modelised as a sphere) from the outside. It means we want to maximize the inradius $r(K)$ of a convex $K \in \mathcal{K}$.

The shape optimization problem

We are now interested in the convex combination of the three functionals δ, D_∞ and r :

$$J_{t,u}(K) = tr(K) + uD(K) + (1 - t - u)\delta(K)$$

Goal :

$$\max_{\mathcal{K}_{A,r_0}} J_{t,u}(K)$$

Where $\mathcal{K}_{A,r_0}$ is the set of convex bodies of area A and inradius higher or equal than r_0.

first results

for $u = 1$, we want to maximize the diameter. the solution, G_{A,r_0}, is the convex hull a disc of radius r and two centrally symemetric points at a distance D such that $|G_{A,r_0}| = A$. The result was proved in [1]

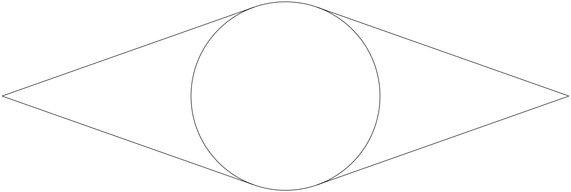


FIGURE: The unique maximizer for $u = 1$

Remarque

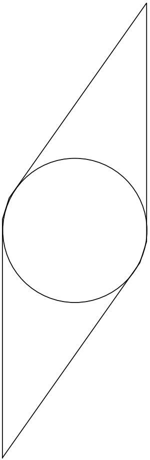
This set is also the minimizer of the area for a given diameter D *and inradius* r*. We denote it by* $G_{D,r}$*.*

first results

for t=1, the solution is the disc of area A.

First results

for $t = u = 0$ the solutions are the tiling domains. But it is interesting to know who are those of maximal diameter.
Answer : it looks like G_{A,r_0}. We denote it by H_{A,r_0}



Sketch of the proof

First, we want to maximize the number of sides of the polygon. We search the solution among the hexagons. Then we show that the hexagon saturates the inradius constraint

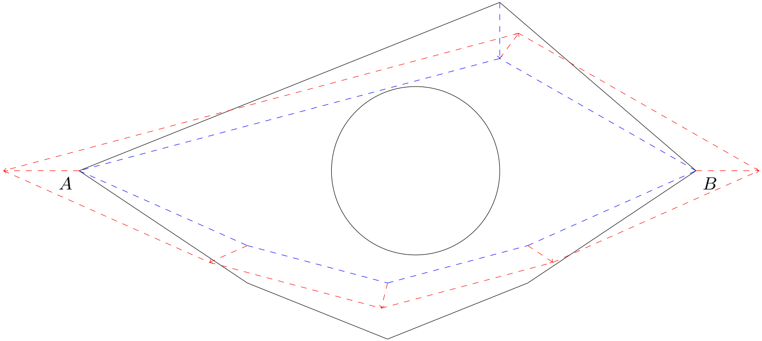


FIGURE: An optimal hexagon saturates the inradius constraint

Sketch of the proof

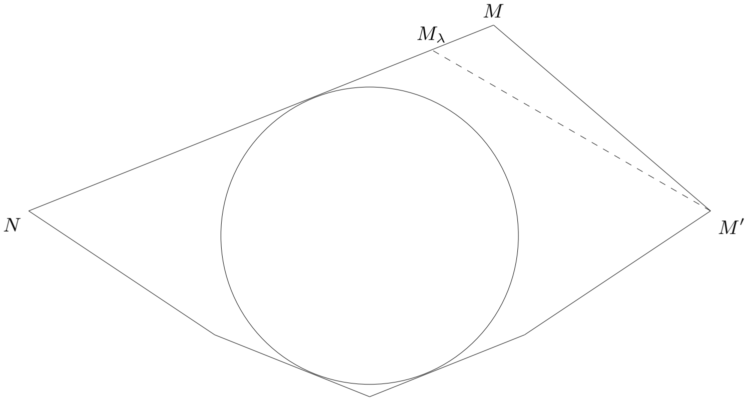


FIGURE: each side is tangent to the inner circle

Sketch of the proof

This allows to give an easy parametrization of the problem.

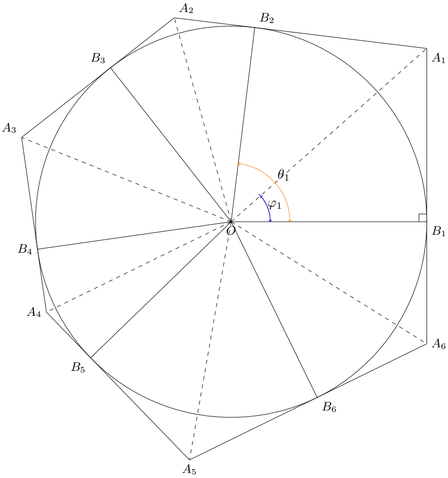


FIGURE: A parametrization of the problem

Sketch of the proof

$$\Theta = \{\Phi = (\varphi_1, ..., \varphi_6) \in [0, \frac{\pi}{2}]^6,\ \sum_{i=1}^{6} \varphi_i = \pi,\ \sum_{i=1}^{6} \tan(\varphi_i) = A\}.$$

The problem finally rewrites :

$$\max_{\Phi \in \Theta} \frac{1}{\cos(\varphi_1)} + \frac{1}{\cos(\varphi_4)}.$$

Solving this problem leads to the hexagon H_{A,r_0}

General Case

for any t and u,

- Fix r and D.
- Find the convex with given inradius r and diameter D with the highest density. This one maximizes $J_{t,u}$ among the convex of given r and D.
- We are able to write the density of such a set as a function of r and d.
- the problem reduces to a 2D optimization problem.

General Case

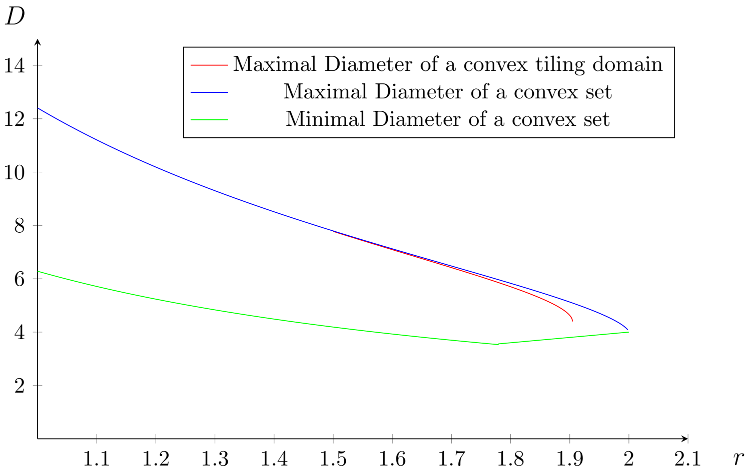


FIGURE: Blashke diagram for r and D with $A = 4\pi$

General Case

Theorem

If $D \geq 4\sqrt{3}r$,

$$\max_{r(K)=r, D(K)=D, |K|=A} \delta(K) = \frac{A}{r\sqrt{D^2 - 4r^2} + \frac{8r^3}{D+\sqrt{D^2-4r^2}}}$$

If $\sqrt{A/2\sqrt{3}} < r \leq \sqrt{A/\pi}$ *and* $D < 4/\sqrt{3}r$,

$$\max_{r(K)=r, D(K)=D, |K|=A} \delta(K) = \frac{A}{2\sqrt{3}r^2},$$

So the initial problem in $\mathcal{K}_{A,r_0}$ *reduces to an optimization problem on* $\mathbb{R}^2$ *with constraints on the variables*

Perspectives

- Finish the study of the 2D functional.
- Investigate the 3D functional, theoretical and numerical aspects.
- Find other criteria by modeling issues that remain to be investigated.
- Explain the folds.

References

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