Computing risk averse equilibrium in incomplete market

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GdR MOA et MIA, France, October, 2017

CERMICS - EPOC

Uncertainty on electricity market

- Today, wholesale electricity markets takes the form of an auction that matches supply and demand
- But, the demand cannot be predicted with absolute certainty. Day-ahead markets must be augmented with balancing ones
- To reduce CO_2 emissions and increase the penetration of renewables, there are increasing amounts of electricity from intermittent sources such as wind and solar
- Equilibrium on the market are then set in a stochastic setting

Multiple equilibrium in a incomplete market

- In Philpott et al. (2013), the authors present a framework for multistage stochastic equilibria
- They show that equilibrium in risk-neutral market and equilibrium in complete risk averse markets can be found as solution of a global optimization problem allowing us to decompose per agent
- What about risk averse equilibrium in incomplete market ?
- We present a toy problem with agreable properties (strong concavity of utility) that displays multiple equilibrium
- Classical computing methods fail to find all equilibria

Outline

Outline

Statement of the problem

- Social planner problem (Optimization problem)
- Equilibrium problem
- Trading risk with Arrow-Debreu securities

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium

Ingredients of the problem

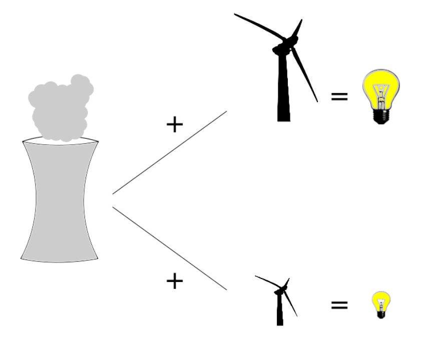

Figure 1: Illustration of the toy problem

- Two time-step market
- One good traded
- Two agents: producer and consumer
- Finite number of scenario $\omega \in \Omega$
- Consumption on second stage only

Producer's welfare

- Step 1: production of x at a marginal cost cx
- Step 2: random production $\mathbf{x}_r$ at uncertain marginal cost $\mathbf{c}_r\mathbf{x}_r$

$$\underbrace{\boldsymbol{W_p}(\omega)}_{\text{producer's welfare}} = - \underbrace{\frac{1}{2}cx^2}_{\text{cost step 1}} - \underbrace{\frac{1}{2}\mathbf{c}_r(\omega)\mathbf{x}_r(\omega)^2}_{\text{cost step 2}}$$

- Step 1: no consumption $\varnothing$
- Step 2: random consumption $\mathbf{y}$ at marginal utility $\mathbf{V} - \mathbf{ry}$

$$\underbrace{\boldsymbol{W_c}(\omega)}_{\text{consumer's welfare}} = \underbrace{\mathbf{V}(\omega)\mathbf{y}(\omega) - \frac{1}{2}\mathbf{r}(\omega)\mathbf{y}(\omega)^2}_{\text{consumer's utility at step 2}}$$

Social planner's welfare

The welfare of the social planner can be defined by

$$\underbrace{W_{sp}(\omega)}_{\text{Social planner's welfare}} = \underbrace{W_p(\omega)}_{\text{Producer's welfare}} + \underbrace{W_c(\omega)}_{\text{Consumer's welfare}}$$

Optimization and uncertainty

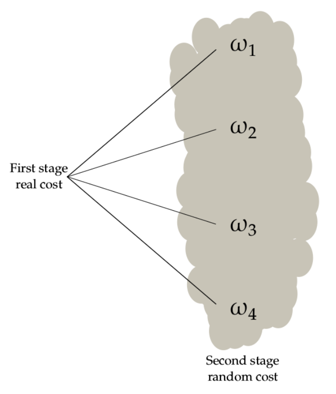

Figure 2: Aggregating uncertainty with a risk measure to obtain real value

To be able to do optimization, we aggregate uncertainty using:

- the expectation $\mathbb{E}_{\mathbb{P}}$: risk neutral
- a risk measure $\mathbb{F}$: risk averse

Risk neutral social planner problem

Given a probability distribution $\mathbb{P}$ on Ω, we can define a risk neutral social planner problem

$$\text{RNSP}(\mathbb{P}): \max_{x, \mathbf{x}_r, \mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_{sp}]}_{\text{expected welfare}}$$

$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}} \ , \ \forall \omega \in \Omega$$

Risk averse social planner problem

Given a risk measure $\mathbb{F}$, we can define a risk averse social planner problem

$$\text{RASP}(\mathbb{F}): \max_{x,\mathbf{x}_r,\mathbf{y}} \underbrace{\mathbb{F}[\boldsymbol{W}_{sp}]}_{\text{risk adjusted welfare}}$$

$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}, \quad \forall \omega \in \Omega$$

Coherent risk measures

We study coherent risk measures defined by
(see Artzner et al. (1999))

$$\mathbb{F}[\boldsymbol{Z}] = \min_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\boldsymbol{Z}]$$

where $\mathcal{Q}$ is a convex set of probability distributions over Ω

Risk averse social planner problem with polyhedral risk measure

- If $\mathbb{Q}$ is a polyhedron defined by K extreme points $(\mathbb{Q}_k)_{k \in [\![1;K]\!]}$, then the risk measure $\mathbb{F}$ is said to be polyhedral and is defined by

$$\mathbb{F}[\boldsymbol{Z}] = \min_{\mathbb{Q}_1,\dots,\mathbb{Q}_K} \mathbb{E}_{\mathbb{Q}_k}[\boldsymbol{Z}]$$

- The problem RASP($\mathbb{F}$) where $\mathbb{F}$ is polyhedral can be written in a more convenient form for optimization

$$\begin{aligned}
&\max_{\theta, x, \mathbf{x}_r, \mathbf{y}} \theta \\
&\text{s.t. } \theta \le \mathbb{E}_{\mathbb{Q}_k}[\boldsymbol{W}_{sp}] \ , \quad k \in [\![1;K]\!] \\
&\qquad x + \mathbf{x}_r(\omega) = \mathbf{y}(\omega) \ , \quad \forall \omega \in \Omega
\end{aligned}$$

Statement of the problem

Social planner problem (Optimization problem)

Equilibrium problem

Trading risk with Arrow-Debreu securities

Agent are price takers

Definition

An agent is *price taker* if she acts as if she has no influence on the price.

In the remain of the presentation, we consider that agents are price takers

Definition risk neutral equilibrium

Definition ((See Arrow and Debreu (1954) or Uzawa (1960)))

Given a probability $\mathbb{P}$ on Ω, a risk neutral equilibrium RNEQ($\mathbb{P}$) is a set of prices $\{\boldsymbol{\pi}(\omega)\ ,\ \omega \in \Omega\}$ such that there exists a solution to the system

$$\begin{aligned}
\mathrm{RNEQ}(\mathbb{P}):\ & \max_{x,\mathbf{x}_r} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{expected profit}} \\
& \max_{\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\,\mathbf{y}\right]}_{\text{expected utility}} \\
& \underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}\ , \quad \forall \omega \in \Omega
\end{aligned}$$

Remark on complementarity constraints

- Complementarity constraints are defined by

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0 \ , \ \ \forall \omega \in \Omega$$

- If $\boldsymbol{\pi} > 0$ then supply $=$ demand
- If $\boldsymbol{\pi} = 0$ then supply $\geq$ demand

Definition of risk averse equilibrium

Definition

Given two risk measures $\mathbb{F}_p$ and $\mathbb{F}_c$, a risk averse equilibrium RAEQ$(\mathbb{F}_p, \mathbb{F}_c)$ is a set of prices $\{\boldsymbol{\pi}(\omega) : \omega \in \Omega\}$ such that there exists a solution to the system

$$\mathrm{RAEQ}(\mathbb{F}_p, \mathbb{F}_c): \quad \max_{x, \mathbf{x}_r} \quad \underbrace{\mathbb{F}_p\left[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\right]}_{\text{risk adjusted profit}}$$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\left[\boldsymbol{W}_c - \boldsymbol{\pi}\,\mathbf{y}\right]}_{\text{risk adjusted consumption}}$$

$$\underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}, \quad \forall \omega \in \Omega$$

- If $\mathbb{F}_p = \mathbb{F}_c$ then we write RAEQ$(\mathbb{F})$

Consumer is insensitive to the choice of risk measure

Assuming that the risk measure $\mathbb{F}_c$ of the consumer is monotonic, she can optimize scenario per scenario as she has no first stage decision

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{risk adjusted consumption}}$$

$$\Updownarrow$$

$$\forall \omega \in \Omega\,, \quad \max_{\mathbf{y}(\omega)} \quad \underbrace{\boldsymbol{W}_c(\omega) - \boldsymbol{\pi}(\omega)\mathbf{y}(\omega)}_{\text{scenario independant}}$$

Risk averse equilibrium with polyhedral risk measure

If the risk measure $\mathbb{F}$ is polyhedral, then RAEQ($\mathbb{F}$) reads

$$\text{RAEQ:} \max_{\theta, x, \mathbf{x}_r} \quad \theta$$

$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\right], \ \forall k \in [\![1; K]\!]$$

$$\max_{\mathbf{y}(\omega)} \quad \boldsymbol{W}_c(\omega) - \boldsymbol{\pi}\mathbf{y}(\omega), \ \forall \omega \in \Omega$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0, \ \forall \omega \in \Omega$$

Definition of an Arrow-Debreu security

Definition

An *Arrow-Debreu security* for node $\omega \in \Omega$ is a contract that charges a price $\mu(\omega)$ in the first stage, to receive a payment of 1 in scenario ω.

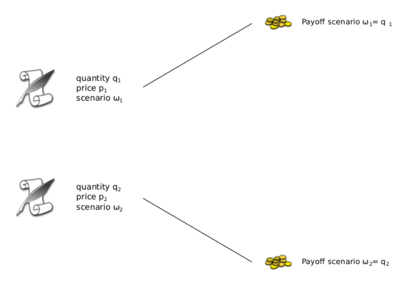

Figure 3: Representation of two Arrow-Debreu securities with two scenarii

Risk averse equilibrium with risk trading

A *risk trading equilibrium* is sets of prices $\{\boldsymbol{\pi}(\omega)\ ,\ \ \omega \in \Omega\}$ and $\{\boldsymbol{\mu}(\omega)\ ,\ \ \omega \in \Omega\}$ such that there exists a solution to the system:

$$\text{RAEQ-AD:}\quad \max_{\theta, x, \mathbf{x}_r} \quad \theta - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{a}(\omega)}_{\text{value of contracts purchased}}$$

$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r) + \mathbf{a}\right]\ ,\ \ \forall k \in [\![1;K]\!]$$

$$\max_{\phi, \mathbf{y}} \quad \phi - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{b}(\omega)}_{\text{value of contracts purchased}}$$

$$\text{s.t.} \quad \phi \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y} + \mathbf{b}\right]\ ,\ \ \forall k \in [\![1;K]\!]$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

$$\underbrace{0 \leq -\mathbf{a}(\omega) - \mathbf{b}(\omega)}_{\text{"supply} \geq \text{demand"}} \perp \boldsymbol{\mu}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

Conclusion

Until now, we have seen

- social planner's problem in risk neutral and risk averse setting
- equilibrium problem in risk neutral and risk averse setting
- risk trading equilibrium problem in risk averse setting

We will study the link between

- risk neutral social planner and equilibrium problem (RNSP and RNEQ)
- risk averse social planner and risk trading equilibrium (RASP and RAEQ-AD)

Outline

Statement of the problem

Links between optimization problems and equilibrium problems

- In the risk neutral case
- In the risk averse case

Multiple risk averse equilibrium

Links between optimization problems and equilibrium problems

- In the risk neutral case
- In the risk averse case

RNSP($\mathbb{P}$) is equivalent to RNEQ($\mathbb{P}$)

Proposition

Let $\mathbb{P}$ be a probability measure over Ω.
The elements $(x^\star \mathbf{x}_r^\star, \mathbf{y}_r^\star)$ are optimal solutions to RNSP($\mathbb{P}$) if and only if there exist non trivial equilibrium prices $\boldsymbol{\pi}$ for RNEQ($\mathbb{P}$) with associated optimal controls $(x^\star, \mathbf{x}_r^\star, \mathbf{y}^\star)$

Corollary

If producer's criterion and consumer's criterion are strictly concave, then RNSP($\mathbb{P}$) admit a unique solution and RNEQ($\mathbb{P}$) admit a unique equilibrium.

Links between optimization problems and equilibrium problems

In the risk neutral case

In the risk averse case

RAEQ-AD is equivalent to RASP

We adapt a result of Ralph and Smeers (2015)

Proposition

Suppose given equilibrium prices $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ such that the finite valued vector $(x, \mathbf{x}_r, \mathbf{y}, \mathbf{a}, \mathbf{b}, \theta, \varphi)$ solves RAEQ-AD($\mathbb{F}$). Then $\boldsymbol{\pi}$ are equilibrium price for RNEQ(μ) with optimal value vector $(x, \mathbf{x}_r, \mathbf{y})$. Moreover, $(x, \mathbf{x}_r, \mathbf{y})$ solves RASP($\mathbb{F}$) where μ is the worst case probability.

The reverse holds true

Summing up equivalences

- We have shown two equivalences

$$\text{RNSP}(\mathbb{P}) \Leftrightarrow \text{RNEQ}(\mathbb{P}) \,, \qquad \text{(risk neutral setting)}$$
$$\text{RASP}(\mathbb{F}) \Leftrightarrow \underbrace{\text{RAEQ-AD}(\mathbb{F})}_{\text{complete market}} \,, \qquad \text{(risk averse setting)}$$

  that lead to result about uniqueness of equilibrium and methods of decomposition

- What can we say about $\underbrace{\text{RAEQ}(\mathbb{F})}_{\text{incomplete market}}$?

Outline

Multiple risk averse equilibrium

Numerical results

Analytical results

Recall on the problem

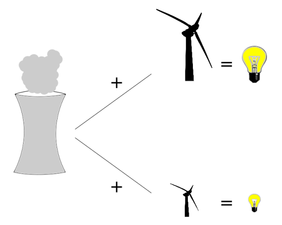

Figure 4: Illustration of the toy problem

Recall:

- Two time-step market
- One good traded
- Two agents
- Consumption on second stage only

We focus on:

- Two scenarios ω_1 and ω_2
- Two prices: π_1 and π_2
- Five controls: x, x_1, x_2, y_1 and y_2
- Two probabilities $(\underline{p}, 1 - \underline{p})$ and $(\bar{p}, 1 - \bar{p})$
- $\underline{p} = \frac{1}{4}$, $\bar{p} = \frac{3}{4}$
- prices $0 < \pi_1 < \pi_2$

Computing an equilibrium with GAMS

- GAMS with the solver PATH in the EMP framework
  (See Britz et al. (2013), Brook et al. (1988), Ferris and Munson (2000) and Ferris et al. (2009))

- different starting points defined by a grid 100×100 over the square $[1.220; 1.255] \times [2.05; 2.18]$

- We find one equilibrium defined by

$$\boldsymbol{\pi} = (\pi_1, \pi_2) = (1.23578; 2.10953)$$

Walras's tâtonnement algorithm (See Uzawa (1960))

Then we compute the equilibrium using a tâtonnement algorithm

Data: MAX-ITER, $(\pi_1^0, \pi_2^0), \tau$

Result: A couple $(\pi_1^\star, \pi_2^\star)$ approximating equilibrium price $\pi_\sharp$

for *k from* 0 *to MAX-ITER* **do**

Compute an optimal decision for each player given a price :

$$x\ ,\ \ x_1, x_2 = \arg\max \mathbb{F}\big[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\big];$$

$$y(\omega) = \arg\max \mathbb{F}[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}];$$

Update the price :

$$\pi_1 = \pi_1 - \tau \max\big\{0; y_1 - (x + x_1)\big\};$$

$$\pi_2 = \pi_2 - \tau \max\big\{0; y_2 - (x + x_2)\big\};$$

end

return (π_1, π_2)

Algorithm 1: Walras' tâtonnement

Computing equilibria with Walras's tâtonnement

- Running Walras's tâtonnement algorithm starting from (1.25; 2.06), respectively from (1.22; 2.18), with 100 iterations and a step size of 0.1, we find two new equilibria

$$\boldsymbol{\pi} = (1.2256; 2.0698) \text{ and } \boldsymbol{\pi} = (1.2478; 2.1564)$$

- An alternative tatônnement method called FastMarket (see Facchinei and Kanzow (2007)) find the same equilibria

Summing up about computing equilibrium

	Equilibrium prices	Risk adjusted welfares
red (Tâtonnement)	$(1.2478; 2.1564)$	$(2.113; 0.845)$
blue (GAMS)	$(1.2358; 2.1095)$	$(2.134; 0.821)$
green (Tâtonnement)	$(1.2256; 2.0698)$	$(2.152; 0.798)$

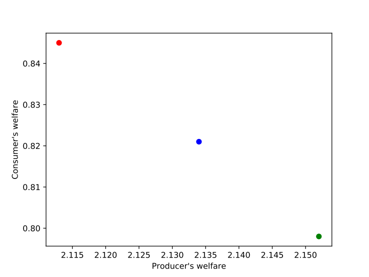

Figure 5: Representation of equilibrium in terms of welfare

- No equilibrium dominates an other

Multiple risk averse equilibrium

Numerical results

Analytical results

Optimal control of agents with respect to a price π

There are three regimes

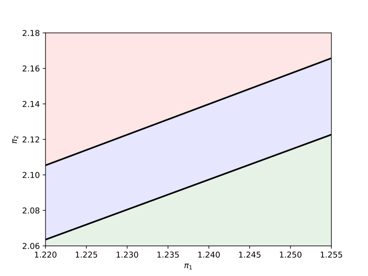

Figure 6: Illustration of the three regimes

condition	$x^\sharp$	$x_i^\sharp$	$y_i^\sharp$
$x_c \leq \frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$
$\frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c} \leq x_c \leq \frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c}$	x_c	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$
$\frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c} \leq x_c$	$\frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$

Table 1: Optimal control for producer and consumer problems

$$\text{where } x_c(\boldsymbol{\pi}) = \frac{1}{2(\pi_1 - \pi_2)} \left(\frac{\pi_2^2}{2c_2} - \frac{\pi_1^2}{2c_1} \right)$$

Excess production function

We are now looking for prices (π_1, π_2) such that the complementarity constraints are satisfied

$$z_i(\boldsymbol{\pi}) = \underbrace{x^\sharp(\boldsymbol{\pi}) + x_i^\sharp(\boldsymbol{\pi}) - y_i^\sharp(\boldsymbol{\pi}) = 0}_{\text{market clears for optimal control}}, \qquad i \in \{1, 2\}$$

This excess functions have three regime. In the green and red part the equation is linear, in the blue part the equation is quadratic.

Representation of analytical solutions (scenario 1)

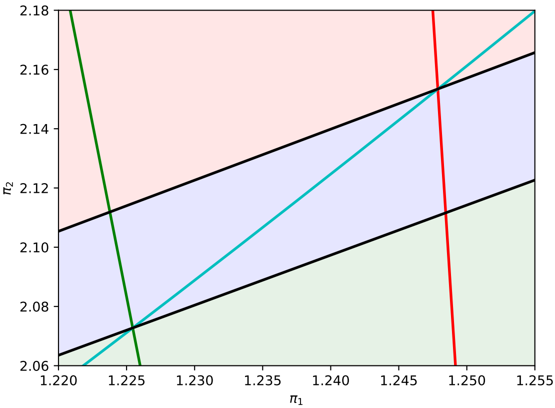

Figure 7: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (scenario 2)

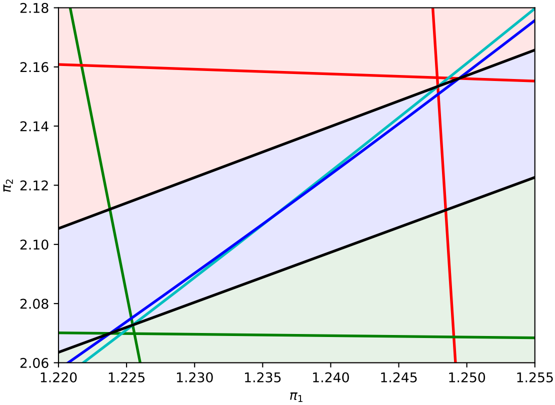

Figure 8: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (red equilibrium)

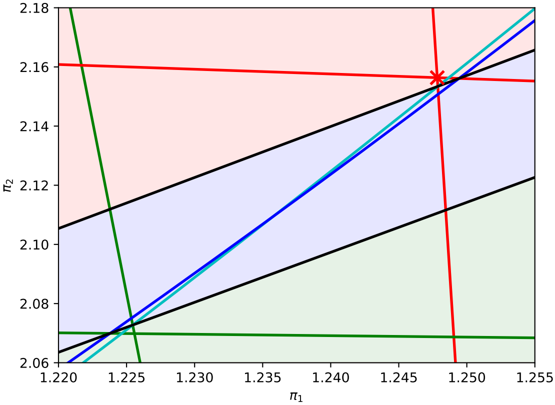

Figure 9: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (blue equilibrium)

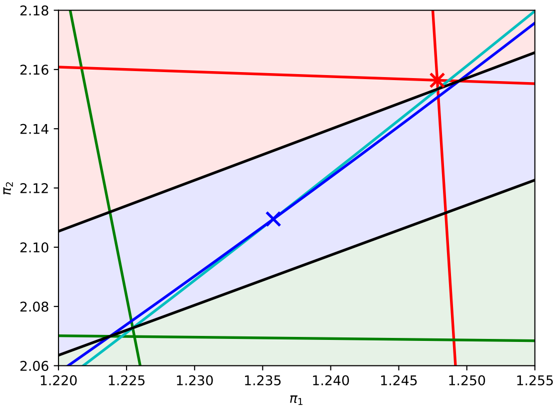

Figure 10: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (green equilibrium)

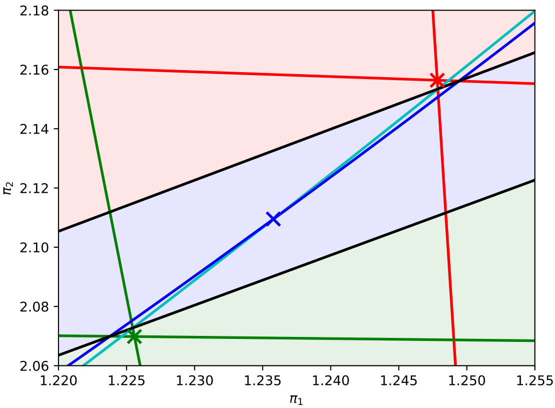

Figure 11: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Some interesting remarks

Remark

The PATH solver find the blue equilibrium, while the tatônnements methods find equilibrium green and red. Interestingly it can be shown that the blue equilibrium is unstable in the sense that the dynamical system driven by $\pi' = z(\pi)$ is unstable around the blue equilibrium.

Remark

There exists a set of non-zero measure of parameters $V_1, V_2, c, c_1, c_2, r_1$, and r_2 (albeit small), that have three distinct equilibrium with the same properties.

Remark

We can show that the blue equilibrium is a convex combination of red and green equilibrium.

Stability of equilibriums (red equilibrium)

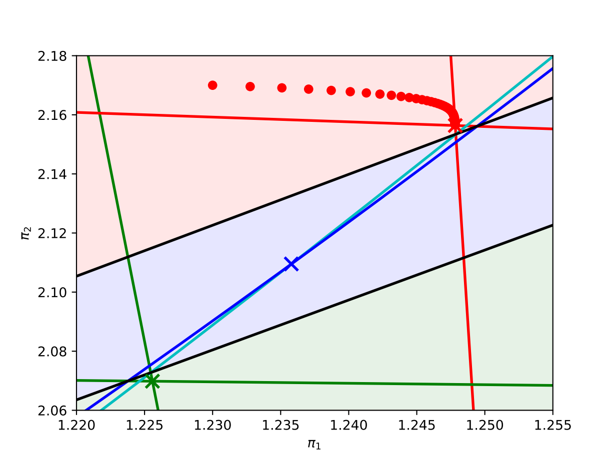

Figure 12: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (blue equilibrium)

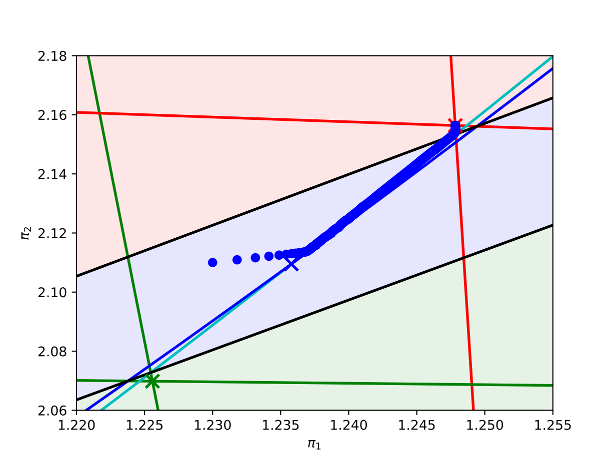

Figure 13: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (green equilibrium)

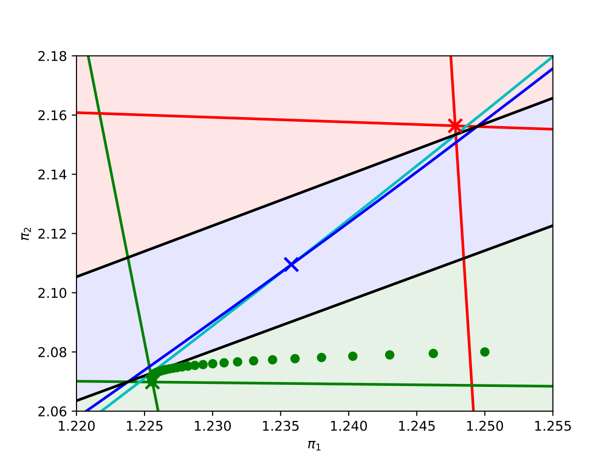

Figure 14: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (vector field)

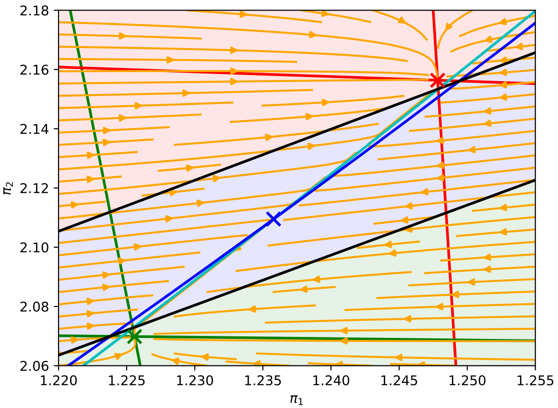

Figure 15: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Conclusion

In this talk we have

- shown an equivalence between risk averse social planner problem
  and risk trading equilibrium (respectively risk neutral equivalence)
- given theorems of uniqueness of equilibrium
- shown non uniqueness of equilibrium in risk averse setting without Arrow-Debreu securities

On going work

- Does the counter example extend with multiple agents and scenarios ?
- Do we have uniqueness with bounds on the number of Arrow-Debreu securities exchanged ?

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