Prox-regularity and generalized equations

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Based on a joint work with Samir Adly (Limoges) and Lionel Thibault (Montpellier),

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Outline

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Notation

- All vector spaces will be real vector spaces.
- For X a (real) normed space, $x \in S \subset X$, one sets:
  - *The distance function from S to x*

$$d_S(x) =: d(x,S) := \inf_{y \in S} \|x - y\|.$$

  - The *nearest points* of x in S

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

When $\mathrm{Proj}_S(x)$ contains one and only one vector $y \in X$, we set $\mathrm{proj}_S(x) := y$.

Definition of uniform prox-regularity

For any $\emptyset \neq S \subset \mathscr{H}$ and any $r \in]0, +\infty]$, one sets $U_r(S) := \{x \in \mathscr{H} : d_S(x) < r\}$.

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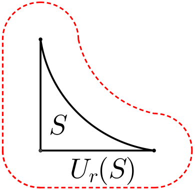

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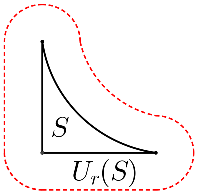

Definition

Let S be a nonempty closed subset of a Hilbert space $\mathscr{H}$ and $r \in]0, +\infty]$ be an extended real. One says that S is *r-prox-regular* (or *uniformly prox-regular with constant r*) whenever the mapping $\mathrm{proj}_S : U_r(S) \to \mathscr{H}$ is well-defined and norm-to-norm continuous.

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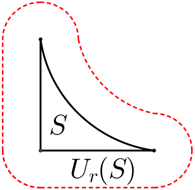

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Notable contributors: H. Federer (1959); J.-P. Vial (1983); A. Canino (1988); A. Shapiro (1994); F.H. Clarke, R.L. Stern, P.R. Wolenski (1995); R.A. Poliquin, R. T. Rockafellar, L. Thibault (2000).

Proximal normal cone

Definition

Let S be a subset of a Hilbert space $\mathscr{H}$. One defines the *proximal normal cone to S at $x \in S$* as the set

$$N(S;x) := \{v \in \mathscr{H} : \exists\, r > 0,\ x \in \mathrm{Proj}_S(x + rv)\}.$$

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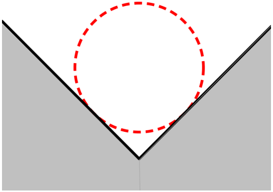


Figure: N is often reduced to 0

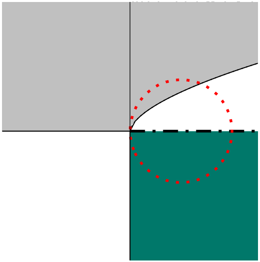


Figure: N fails to be closed.

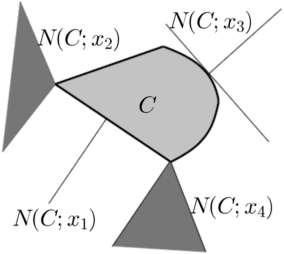


Figure: $N(C;\cdot)$ for a convex set C.

Characterizations of prox-regularity

Theorem

Let S be a nonempty closed subset of a Hilbert space $\mathscr{H}$, $r \in]0,+\infty]$. **Are equivalent:**

(a) S is r-prox-regular;

(b) The mapping proj_S is well-defined on $U_r(S)$ and

$$\|\mathrm{proj}_S(u) - \mathrm{proj}_S(v)\| \leq \left(1 - \frac{d_S(u)}{2r} - \frac{d_S(v)}{2r}\right)^{-1} \|u - v\|;$$

(c) For all $x_1, x_2 \in S$, for all $v_1 \in N(S;x_1) \cap \mathbb{B}_{\mathscr{H}}$, for all $v_2 \in N(S;x_2) \cap \mathbb{B}_{\mathscr{H}}$,

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2;$$

(d) For all $x, y \in S$ and all $t \in [0,1]$ with $tx + (1-t)y \in U_r(S)$,

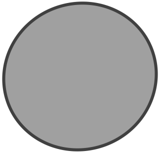
$$d_S(tx + (1-t)y) \leq \frac{1}{2r} t(1-t) \|x - y\|^2;$$

(e) The function d_S^2 is $C^{1,1}$ on $U_r(S)$ and $\nabla d_S^2(u) = 2(u - \mathrm{proj}_S(u))$ for all $u \in U_r(S)$.

If S is weakly closed, then one can add the following to the list of equivalences:

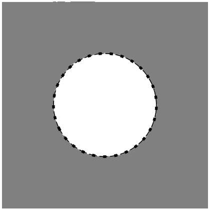
(f) proj_S is well-defined on $U_r(S)$.

Prox-regular sets - examples and counter-examples

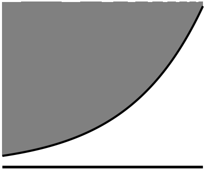


Nonempty closed convex
$\Leftrightarrow$ ∞-prox-regular

Lack of prox-regularity ("angle")



$\mathscr{H} \setminus B(0, r)$ is r-prox-regular



Lack of prox-regularity ("crushing")

Well-posedness of Moreau sweeping process

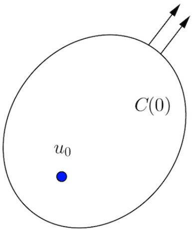
Theorem (J.J. Moreau-1971)

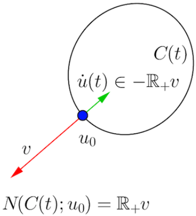
Let $C : I = [0,T] \rightrightarrows \mathscr{H}$ be a nonempty closed **convex**-valued multimapping. Assume that there exists a **nondecreasing absolutely continuous function** $v : I \to [0,+\infty[$ such that

$$|d(y,C(t)) - d(y,C(s))| \le v(t) - v(s) \quad \text{for all } y \in \mathscr{H}, s,t \in I \text{ with } s \le t.$$

Then, for each $u_0 \in C(0)$, there exists one and only one **absolutely continuous mapping** $u : I \to \mathscr{H}$ such that

$$\begin{cases} -\dot{u}(t) \in N(C(t);u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) & \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$





Idea of the proof

Proof 1. Existence of solutions based on Moreau's catching-up algorithm.

Time discretization $0 = t_0^n < \ldots < t_{p(n)}^n = T$ + iterations of the form $u_i^n = \mathrm{proj}_{C(t_i^n)}(u_{i-1}^n)$ (with $u_0^n := u_0$ where u_0 is the initial condition) + suitable interpolation $\Rightarrow$ Sequence of mappings $(u_n(\cdot))$ $\Rightarrow$ Convergence to a solution $u(\cdot)$.

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Proof 2. Regularization of normal cone.

For each $\lambda > 0$, $u_\lambda(\cdot)$ solution of (ODE)

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(0) = u_0 \end{cases}$$

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$\hookrightarrow$ Well-posedness of Moreau's sweeping process holds replacing "closed convex valued" by r-prox-regular valued for some $r \in]0, +\infty]$.

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Theorem (Continuous selection, E. Michael-1956)

Let X be a metric space, E be a Banach space and $F : X \rightrightarrows E$ be a lower semicontinuous multimapping with **nonempty closed convex values**.
Then, there exists a **continuous selection** for F (i.e., a continuous mapping $f : X \to E$ with $f(x) \in F(x)$ for every $x \in X$).

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Proposition

Let $r > 0$ be a real, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a Hausdorff continuous multimapping with r-prox-regular values and let $\eta \in [0, 1[$ such that

$$\frac{n}{2(n+1)} \operatorname{diam} F(x) \leq r^2 \eta \quad \text{for all } x \in \mathbb{R}^n.$$

Then, there exists a continuous selection for F.

Some other applications

- **(P.D.E.)** Viscosity solution for Hamilton-Jacobi equations.

- **(Optimal control)** Prox-regularity of minimum time function (P. Cannarsa & C. Sinestrari (1995),...).

- **(Spectral theory)** Prox-regularity of spectral functions/sets (A.S. Lewis (96, 99), A.S. Lewis, J. Malick & A. Daniilidis (2008),...).

- **(Measure and geometry)** Extension of isoperimetric inequality (G. Colombo & T.K. Nguyen (2009),...).

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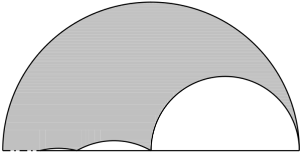
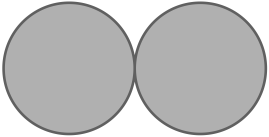
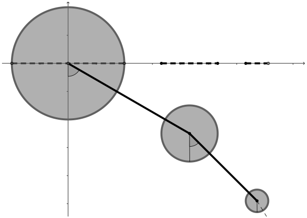
Prox-regularity and preservation: counter-examples



The intersection of prox-regular sets fails to be prox-regular



Non prox-regular union of two convex sets



The projection along a vector space of a prox-regular set fails to be prox-regular

Non prox-regular (sub)-level set

Preservation: state of the art I

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  - Study of inverse image under the condition

$$d(x, F^{-1}(D)) \leq \gamma d(F(x), D).$$

  - Counter-example/study of the intersection under the condition

$$d(x, \bigcap_{k=1}^{m} S_k) \leq \gamma \sum_{k=1}^{m} d(x, S_k).$$

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- G. Colombo and L. Thibault (2010):
  - Study of the inverse image under the condition
$$N(F^{-1}(D); x) \cap \mathbb{B} \subset DF(x)^{\star}(N(D; F(x)) \cap \gamma\mathbb{B}).$$
  - Study of the intersection under the condition
$$N(\bigcap_{k=1}^{m} S_k; x) \cap \mathbb{B} \subset N(S_1; x) \cap \gamma\mathbb{B} + \ldots + N(S_m; x) \cap \gamma\mathbb{B}.$$

Normal cone inverse image property

Proposition

Let $\mathscr{H}, \mathscr{H}'$ be two Hilbert spaces, $g : \mathscr{H} \to \mathscr{H}'$ be a differentiable mapping on $\mathscr{H}$ and S be a r-prox-regular subset of $\mathscr{H}$ for some $r \in]0, +\infty]$. **Assume that:**

(i) *there exists $K > 0$ such that g and Dg are K-Lipschitz continuous on $\mathscr{H}$;*

(ii) *there exists $\beta > 0$ such that for every $x \in g^{-1}(S)$,*

$$N(g^{-1}(S); x) \cap \mathbb{B}_{\mathscr{H}} \subset Dg(x)^{\star}(N(S; g(x)) \cap \beta \mathbb{B}_{\mathscr{H}'}).$$

Then, *the set $g^{-1}(S)$ is $\frac{r}{\beta K(K+r)}$-prox-regular.*

Preservation: state of the art II

- S. Adly, N., L. Thibault (2016)

Sufficient conditions guaranteeing the prox-regularity for:

- A set defined by equality constraints

$$C = \{x \in \mathscr{H} : G(x) = 0\},$$

with $G : \mathscr{H} \to Y$ under an openness condition $s\mathbb{B} \subset DG(x)(\mathbb{B})$.

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- A set defined by inequality/equality constraints $g_i : \mathscr{H} \to \mathbb{R}$

$$\{x \in \mathscr{H} : g_1(x) \leq 0, \dots, g_m(x) \leq 0, g_{m+1}(x) = 0, \dots, g_{m+n}(x) = 0\}$$

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- The intersection of two prox-regular sets S_1, S_2 under an openness condition

$$s\mathbb{B}_{\mathscr{H}} \subset T(S_1; x_1) \cap \mathbb{B}_{\mathscr{H}} - T(S_2; x_2) \cap \mathbb{B}_{\mathscr{H}},$$

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- Inverse image of a prox-regular set $G^{-1}(D)$ with $G : \mathscr{H} \to \mathscr{H}'$ under the openness condition

$$s\mathbb{B}_{\mathscr{H}'} \subset DG(x)(\mathbb{B}_{\mathscr{H}}) - T(D; G(x) - y).$$

Nonsmooth inequality constraints

Theorem (S. Adly, N., L. Thibault (2016))

Let $\mathscr{H}$ be a Hilbert space, $g_1, \dots, g_m : \mathscr{H} \to \mathbb{R}$ such that

$$C = \{x \in \mathscr{H} : g_1(x) \leq 0, \dots, g_m(x) \leq 0\} \neq \emptyset.$$

Assume that there exists $\rho \in]0, +\infty]$ such that:

(i) for each $k \in \{1, \dots, m\}$, g_k is continuous on $U_\rho(C)$;
(ii) there exists $\gamma \geq 0$ such that for all $k \in \{1, \dots, m\}$, for all $x_1, x_2 \in U_\rho(C)$, for all $v_1 \in \partial_C g_k(x_1)$ and for all $v_2 \in \partial_C g_k(x_2)$

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\gamma \|x_1 - x_2\|^2.$$

Assume also that there exists $\delta > 0$ such that for all $x \in \operatorname{bd} C$, there exists $\overline{v} \in \mathbb{B}_{\mathscr{H}}$ satisfying for all $k \in \{1, \dots, m\}$ and for all $\xi \in \partial_C g_k(x)$,

$$\langle \xi, \overline{v} \rangle \leq -\delta \qquad \text{(S.U.)}$$

Then, C is r-prox-regular with $r = \min\left\{\rho, \frac{\delta}{\gamma}\right\}$.

(i) and (ii) $\Leftrightarrow$ g_k continuous and semiconvex on $U_\rho(C)$ (i.e., $g_k = f_k - C\|\cdot\|^2$ for some $C \geq 0$, f_k convex).

Interpretation of uniform Slater's condition

$$C := \{x \in \mathscr{H} : g_1(x) \leq 0, \ldots, g_m(x) \leq 0\}.$$

(S.U.) $\exists \delta > 0, \forall x \in \mathrm{bd}C, \exists \overline{v}_x \in \mathbb{B}, \forall k \in \{1, \ldots, m\}, \forall \xi \in \partial_C g_k(x), \langle \xi, \overline{v}_x \rangle \leq -\delta$

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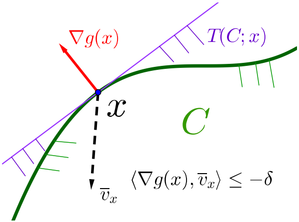
1. $0 \in \partial_C g_k(x)$ for some $x \in \mathrm{bd}\, C \Rightarrow$ Condition $(S.U.)$ does not hold.
2. Case of one smooth constraint: $g \Rightarrow C = \{g \leq 0\}$ and $\partial_C g(x) = \{\nabla g(x)\}$.

$$\hookrightarrow \quad \exists \delta > 0, \forall x \in C \text{ with } g(x) = 0, \exists \overline{v}_x \in \mathbb{B}, \langle \nabla g(x), \overline{v}_x \rangle \leq -\delta.$$



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where $f : \mathscr{H} \to \mathscr{H}'$ is a (single)-valued mapping and $F : \mathscr{H} \rightrightarrows \mathscr{H}'$ is a multimapping.

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  - **Example 1.** A set of constraints can be written as a solution set of a generalized equation:

$$\{x \in \mathscr{H} : f_1(x) \leq 0, \dots, f_m(x) \leq 0, f_{m+1}(x) = 0, \dots, f_{m+n}(x) = 0\} = \{x \in \mathscr{H} : 0 \in f(x) + F(x)\}$$

where $f := (f_1, \dots, f_{m+n})$ and $F :\equiv \mathbb{R}^m_+ \times \{0_{\mathbb{R}^n}\}$.

  - **Example 2.** An intersection of sets is a solution set of a generalized equation:

$$\bigcap_{i=1}^{m} S_i = \left\{ x \in \mathscr{H} : 0 \in (-x, -x) + \prod_{i=1}^{m} S_i \right\}.$$

Metric regularity (definition)

Recall that for a multimapping $M : X \rightrightarrows Y$, the *graph* of M is the set

$$\operatorname{gph} M := \{(x, y) \in X \times Y : y \in M(x)\}$$

and the *inverse image* of a given $\overline{y} \in Y$ is

$$M^{-1}(\overline{y}) := \{x \in X : \overline{y} \in M(x)\}.$$

Definition

Let X, Y be two normed spaces, $M : X \rightrightarrows Y$ be a multimapping, $(\overline{x}, \overline{y}) \in \operatorname{gph} M$. One says that M is *metrically regular at* $\overline{x}$ *for* $\overline{y}$ provided there exist $\gamma \geq 0$ and neighborhoods U and V of $\overline{x}$ and $\overline{y}$ such that

$$d(x, M^{-1}(y)) \leq \gamma d(y, M(x)) \quad \text{for all } (x, y) \in U \times V.$$

• Concept of "metric regularity" goes back to Banach open mapping theorem $\sim$ 1930 (term "metric regularity" coined by J.M. Borwein (1986)).

Robinson-Ursescu theorem (1975-1976)

$\operatorname{gph} M$ is convex $\Leftrightarrow tM(x) + (1-t)M(x') \subset M(tx + (1-t)x')\ \forall x, x' \in X, \forall t \in [0,1]$.

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Theorem

Let X, Y be Banach spaces, $M : X \rightrightarrows Y$ be a multimapping with closed convex graph, $(\overline{x},\overline{y}) \in \operatorname{gph} M$. **Assume that** there exists $c > 0$ such that

$$\overline{y} + c\mathbb{U}_Y \subset M(\overline{x} + \mathbb{B}_X).$$

Then, for every $x \in X$ and every $y \in \overline{y} + c\mathbb{U}_Y$, one has

$$d(x, M^{-1}(y)) \leq (c - \|y - \overline{y}\|)^{-1}(1 + \|x - \overline{x}\|)d(y, M(x)).$$

Further, M is metrically regular at $\overline{x}$ for $\overline{y}$ if and only if $\overline{y} \in \operatorname{int} M(X)$; in such a case the latter inequality holds for some $c > 0$.

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- A. Jourani (1996); H. Huang et R.X. Li (2011): paraconvex multimapping M.
- X.Y. Zheng et K.F. Ng (2012): multimapping M with locally prox-regular graph.
- X.Y. Zheng, Q.H. He (2014): locally subsmooth graph.

Robinson-Ursescu and prox-regularity

Theorem (S. Adly, N., L. Thibault (2017))

Let X, Y be Banach spaces, $M : X \rightrightarrows Y$ be a multimapping with closed graph, Q be a nonempty subset of $\mathrm{gph}\, M$.

Assume that:

(i) the set $\mathrm{gph}\, M$ is r-prox-regular for some $r \in]0, +\infty]$.

(ii) there exist $\alpha, \beta, \rho \in]0, +\infty[$ with

$$\beta > \frac{3\alpha}{\rho} + \frac{1}{2r}(1 + \frac{1}{\rho})\left(4\alpha^2 + (\beta - \frac{\alpha}{\rho})^2\right)$$

such that for all $(\overline{x}, \overline{y}) \in Q$,

$$\overline{y} + \beta \mathbb{U}_Y \subset M(\overline{x} + \alpha \mathbb{B}_X);$$

Then, there exists a real $\gamma \in [0, \rho[$ such that for every $(\overline{x}, \overline{y}) \in Q$, there exists a real $\delta > 0$ satisfying for all $x \in B(\overline{x}, \delta)$, for all $y \in B(\overline{y}, \delta)$,

$$d(x, M^{-1}(y)) \leq \gamma d(y, M(x)).$$

Prox-regularity and generalized equations

Theorem (S. Adly, N., L. Thibault (2017))

Let $\mathscr{H}, \mathscr{H}'$ be Hilbert spaces, $f : \mathscr{H} \to \mathscr{H}'$ be a mapping and $F : \mathscr{H} \rightrightarrows \mathscr{H}'$ be a multimapping such that $S = \{x \in \mathscr{H} : 0 \in f(x) + F(x)\} \neq \emptyset$.

Assume that:

(i) $\operatorname{gph} F$ is r-prox-regular with $r \in]0, +\infty]$;

(ii) f is differentiable on $\mathscr{H}$ with $Df : \mathscr{H} \to \mathscr{H}'$ γ-Lipschitz on $\mathscr{H}$ with $\gamma \geq 0$ and there exist $\rho \in]0, +\infty]$, $L \geq 0$ such that for all $x, y \in S$ with $\|x - y\| < 2\rho$,

$$\|f(x) - f(y)\| \leq L\|x - y\|;$$

(iii) there exist $\alpha, \beta, s > 0$ with $\beta > \frac{3\alpha}{s} + \frac{1}{2r}(1 + \frac{1}{s})(4\alpha^2 + (\beta - \frac{\alpha}{s})^2)$ such that for all $\overline{x} \in S$,

$$\beta \mathbb{U}_{(\mathscr{H} \times \mathscr{H}')^2} \subset -\{((x,y),(x,y)) : (x,y) \in (\overline{x}, -f(\overline{x})) + \alpha \mathbb{B}_{\mathscr{H} \times \mathscr{H}'}\} + \operatorname{gph} F \times \operatorname{gph}(-f).$$

Then, the set S is r'-prox-regular with $r' = \min\left\{\rho, \frac{\min\{r, \frac{1}{\gamma}\}}{4s(L^2+1)}\right\}$.

Selection of solutions for generalized equations

Let $F : \mathscr{H} \rightrightarrows \mathscr{H}$ be a multimapping, $g : \mathscr{H} \to \mathbb{R}$ a function and $x_0 \in \mathscr{H}$ be a solution of the generalized equation

$$F(x) \ni 0, x \in \mathscr{H}.$$

Proposition

Prox-regularity of $F^{-1}(0) := \{x \in \mathscr{H} : 0 \in F(x)\}$ + Palais-Smale condition $\Rightarrow$ Existence and uniqueness of a solution $x(\cdot)$ of the dynamical system

$$\begin{cases} -\dot{x}(t) \in \nabla g(x(t)) + N(F^{-1}(0); x(t)) \\ x(0) = x_0 \end{cases}$$

with $x(t) \to x_\infty$ such that

$$0 \in \nabla g(x_\infty) + N(F^{-1}(0); x_\infty).$$

Perspectives

- Extend some results to some other class of sets (subsmooth, α-far, etc.).

- Coming back to the problem of intersection of prox-regular sets (number infinite of sets, verifiable conditions, etc.).

- Weak the assumption of the graph prox-regularity in the study of prox-regularity of $\{x \in \mathscr{H} : 0 \in f(x) + F(x)\}$.

- Prox-regularity of solution sets of variational inequality.

- Develop results in the framework of Banach spaces.

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Thank you for your attention