Sensitivity analysis for optimization and optimal control problems

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Generic framework:

\[
\min_x f(x, u); \quad G(x, u) \in K \quad (P_u)
\]

with

- \( U, X, Y \): Banach spaces
- \( K \) closed convex subset of \( Y \)
- \( f : X \times U \to \mathbb{R} \) and \( G : X \times U \to Y \), smooth \((C^\infty)\).
Examples

- $K = \{0\}_{\mathbb{R}^p} \times \mathbb{R}^q$: finitely many equalities and inequalities. If $X = \mathbb{R}^n$: Nonlinear programming.
- **Semi definite programming**: $X = \mathbb{R}^n$ and $K = S^p_-$, with $S^p_-$ set of symmetric matrices of order $p$, or more generally
  
  $$K = S^{p_1}_- \times \cdots S^{p_q}_-$$

- **Semi infinite programming**: $X = \mathbb{R}^n$, $\Omega$ compact metric, $Y = C(\Omega)$, $K = C(\Omega)_-$:
- Optimal control of ODEs
- Optimal control of PDEs
Min \( f(x, u); \quad G(x, u) \in K \) \quad (P_u)

- Set of feasible points: \( F(u) = G^{-1}(\cdot, u)(K) \). Value

\[
\text{val}(u) := \inf \{ f(x, u); \ x \in F(u) \}
\]

Set of solutions

\[
S(u) := \{ x \in F(u); \ f(x, u) = \text{val}(u) \}.
\]
Questions

- Stability of solutions:
  \[ \text{dist}[S(u'), S(u)] = O(\|u' - u\|) \text{ or } O(\|u' - u\|^{1/2}) \]

- Expansion of (approximate) solutions along a path
  \[ u_\tau := \bar{u} + \tau v, \; \tau \in \mathbb{R}_+ \]
  \[ S(u_\tau) = \bar{x} + \tau^\gamma h + o(\tau^\gamma), \; \gamma = 1 \text{ or } 1/2 \]

- Expansion of value function: find subproblems \((L)\) and \((Q)\) such that
  \[ \text{val}(u_\tau) = \text{val}(\bar{u}) + \tau \text{val}(L) + \frac{1}{2} \tau^2 \text{val}(Q) + o(\tau^2). \]
First and second order tangent cones

Nominal problem: \( f(x) = f(x, \bar{u}), \ G(x) = G(x, \bar{u}). \)

Here \( \mathcal{K} \subset X \), possibly nonconvex, e.g. \( \mathcal{K} = G^{-1}(K) \):

- **(Interior) tangent cone**: \( y \in \mathcal{K} \)

\[
T_{\mathcal{K}}(x) = \{ y \in X; \ \text{dist}(x + \tau y, \mathcal{K}) = o(\tau), \ \tau \geq 0 \}
\]

- **(Interior) second order tangent set**: \( x \in \mathcal{K}, \ y \in T_{\mathcal{K}}(x) \).

\[
T^2_{\mathcal{K}}(x, y) = \{ z \in Y; \ \text{dist}(x + \tau y + \frac{1}{2} \tau^2 z, \mathcal{K}) = o(\tau^2), \ \tau \geq 0 \}
\]
Second-order necessary condition

Nominal problem again:

$$\min_x f(x); \quad x \in \mathcal{K} \subset X.$$ 

Use

$$f(x + \tau y + \frac{1}{2} \tau^2 z) = f(x) + \tau Df(x)y + \frac{1}{2} \tau^2 [Df(x)z + D^2 f(x)(y, y)] + o(\tau^2)$$

First-order necessary condition:

$$Df(\bar{x})h \geq 0, \quad \text{for all } h \in T_{\mathcal{K}}(x)$$

Second-order necessary condition: for all $h \in C(\bar{x}) := T_{\mathcal{K}}(x) \cap Df(\bar{x})^\perp$:

$$Df(\bar{x})z + D^2 f(\bar{x})(h, h) \geq 0, \quad \text{for all } z \in T^2_{\mathcal{K}}(x, h)$$
Robinson’s Qualification Condition

Robinson’s qualification condition (RQC) for \( G(x) \in K \):

For some \( \varepsilon > 0 \), \( \varepsilon B \subset G(x) + \text{Im}[DG(x)] - K \) \hspace{1cm} (RQC)

Specific cases:

- Equality constraints: \( K = \{0\} \), reduces to \( DG(x) \) onto
- \( \text{int}(K) \neq \emptyset \), reduces to

  For some \( h \in X \), \( G(x) + DG(x)h \in \text{int}(K) \)

Normal cone

\[
N_K(y) = \{ y^* \in Y^* ; \langle y^*, y' - y \rangle \leq 0, \text{ for all } y' \in K \}
\]
Calculus of first and second order tangent cones

If (RQC) holds then we have

\[ G(x + \tau y + \frac{1}{2}\tau^2 z) = G(x) + \tau DG(x)y + \frac{1}{2}\tau^2[DG(x)z + D^2 G(x)(z,z)] + o(\tau^2) \]

- **Tangent cone**: \( y \in K \)

\[ T_K(x) = \{y \in Y; \ DG(x)y \in T_K(G(x))\} \]

- **Second order tangent set**: \( x \in K, \ y \in T_K(x) \).

\[ T^2_K(x, y) = \{w \in Y^*; \ DG(x)z + D^2 G(x)(y,y) \in T^2_K[G(x), DG(x)y]\} \]
First-order necessary conditions

If (RQC) holds; **Tangent directions**

\[ T_K(\bar{x}) = \{ h \in X; DG(\bar{x})h \in T_K(G(\bar{x})) \} \]

If (RQC) holds; **Critical cone**

\[ C(\bar{x}) = \{ h \perp Df(\bar{x}); \quad DG(\bar{x})h \in T_K(G(\bar{x})) \} \]

**Lagrangian function**: \( L(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle \).

**Set of Lagrange multipliers** non empty and bounded at local solutions

\[ \Lambda(\bar{x}) := \{ \lambda \in N_K(G(\bar{x})); \quad D_x L(\bar{x}, \lambda) = 0 \}. \]

Robinson 76, Zowe and S. Kurcyusz 79
second-order necessary conditions II

(Cominetti 90) **Primal form:**
For all \( h \in C(\bar{x}) \), \( \text{val}(Q_h) \geq 0 \), where
\[
\begin{align*}
Df(\bar{x})z + D^2f(\bar{x})(h, h) & \geq 0 \quad \text{if} \\
DG(\bar{x})z + D^2G(\bar{x})(h, h) & \in T^2_K[G(x), DG(x)h]
\end{align*}
\]

Support function: \( \sigma(\lambda, K) := \sup\{\langle \lambda, w \rangle; \ w \in K\} \)

**Support function of second-order tangent set (Curvature term)**
\[
\Xi(\lambda, h) := \sigma(\lambda, T^2_K[G(\bar{x}), DG(\bar{x})h])
\]

**Lagrangian of \( (Q_h) \):**
\[
D_xL(x, \lambda)z + D^2_xL(x, \lambda)(h, h) - \Xi(\lambda, h).
\]

**Dual form of second-order necessary conditions:**
\[
\max_{\lambda \in \Lambda(\bar{x})} D^2_{xx}L(x, \lambda)(z, z) - \Xi(\lambda, h) \geq 0.
\]
Sign of $\sigma(\lambda, T^2_K(g, \delta g))$

When $\lambda \in N_K(G(\bar{x}))$, $\lambda \perp \delta g$ and $w \in T^2_K(g, \delta g)$:

$$0 \geq \langle \lambda, g + \tau \delta g + \frac{1}{2} \tau^2 w + o(\tau^2) \rangle = \tau^2 [\langle \lambda, w \rangle + o(1)]$$

and so we always have $\Xi(\lambda, h) \leq 0$. In particular, the second-order necessary condition is weaker than the condition

$$\max_{\lambda \in \Lambda(\bar{x})} D^2_{xx} L(x, \lambda)(z, z) \geq 0.$$ 

Cases when $\Xi(\lambda, h) = 0$:

(a) $K$ polyhedron

(b) $K$ pointed cone and $G(\bar{x}) = 0$.

Reduction theory: rewriting of $(P)$ such that (b) holds

(FB and Shapiro 2000)
Example: inequality constraints I

For $K = \mathbb{R}^q$, assuming (RCP) $\equiv$ Mangasarian-Fromovitz

Define sets of active constraints

$$I(x) := \{1 \leq i \leq q; \ G_i(x) = 0\}$$
$$I(x, y) := \{i \in I(x); \ DG_i(x)y = 0\}$$

- **Tangent cone**: $x \in K$,

$$T_K(x) = \{y \in X; \ DG_i(x)z \leq 0, i \in I(x)\}$$

**Set of Lagrange multipliers**

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^p_+; \ \lambda \cdot G(x) = 0; \ Df(\bar{x}) + DG(x)^\top \lambda = 0\}$$
Example: inequality constraints II

- **Second order tangent cone:** \( x \in \mathcal{K}, \ y \in T_{\mathcal{K}}(y) \).

\[
T^2_{\mathcal{K}}(x, y) = \{ z \in X; DG_i(x)z + D^2G_i(x)(y, y) \leq 0, i \in I(x, y) \}
\]

**Second order necessary conditions**

\[
\max_{\lambda \in \Lambda(\bar{x})} D^2_{xx} L(\bar{x}, \lambda)(h, h) \geq 0.
\]
Example: semidefinite programming I

For \( K = S^p_- \), we have

- Qualification: \( G(x) + DG(x)\hat{y} \in S_- \) (negative definite) for some \( \hat{y} \)
- \( E \ p \times s \) matrix whose columns are an orthonormal basis of the kernel of \( A = G(x) \). **Tangent cone**: 
  \[
  T_{S^p_-}(A) = \{ H \in S^p; E^\top HE \preceq 0 \}
  \]

- \( F \ s \times k \) matrix whose columns are an orthonormal basis of the eigenspace associated with the largest eigenvalue of \( E^\top HE \). **Second order tangent cone**: \( x \in K, y \in T_K(y) \).
  
  \[
  T^2_{S^p_-}(A, H) = \{ W \in S^p; F^\top E^\top WEF \preceq 2F^\top E^\top HA^\dagger HEF \}
  \]
  
  with \( A^\dagger \) pseudo-inverse of \( A \).
Example: semidefinite programming II

**Support function of second-order tangent set:** quadratic function

\[
\Xi(\lambda, h) = -h^\top \mathcal{H}(\bar{x}, \lambda) h
\]

with, setting \( G_i(\bar{x}) = \frac{\partial G(\bar{x})}{\partial x_i} \):

\[
[\mathcal{H}(\bar{x}, \lambda)]_{ij} = -2 \lambda \circ \left( G_i(\bar{x})[G(\bar{x})]^\dagger G_j(\bar{x}) \right)
\]

Ref. A. Shapiro, 1997.
Upper estimates: first-order

Generic framework: path \( u_\tau := \bar{u} + \tau v \)

\[
\begin{align*}
\min_x & \quad f(x, u); \quad G(x, u) \in K \\
\bar{x} & \in S(\bar{u}); \text{ Linearized problem} \\
\min_{h \in \mathcal{X}} & \quad Df(\bar{x}, \bar{u})(h, v); \\
DG(\bar{x}, \bar{u})(h, v) & \in T_K[G(\bar{x}, \bar{u})]. \\
\end{align*}
\]

Lagrangian function: \( L(x, u, \lambda) := f(x, u) + \langle \lambda, G(x, u) \rangle \).

If (RQC): \( \text{val}(LP) = \text{val}(LD) \) where

\[
\max_{\lambda \in \Lambda(\bar{x})} \quad DuL(\bar{x}, \bar{u}, \lambda)\nu 
\]

Set of dual solutions: \( S(LD) \). **First-order upper estimate**

\[
\text{val}(u_\tau) \leq \text{val}(\bar{u}) + \tau \text{val}(LP) + o(\tau).
\]
Upper estimates: second-order, pseudo quadratic problem

Pseudo Quadratic problem: primal formulation

$$\text{val}(QP_h) \geq 0,$$

where

$$\begin{align*}
\min_{z \in X} & \quad D_x f(\bar{x}, \bar{u})(z, z) + D^2 f(\bar{x}, \bar{u})(h, v)^2 \\
& \quad D_x G(\bar{x}, \bar{u})z + D^2 G(\bar{x}, \bar{u})(h, v)^2 \\
& \quad \in T^2_K [G(\bar{x}, \bar{u}), DG(\bar{x}, \bar{u})(h, v)].
\end{align*}$$

Curvature term for the perturbation problem

$$\Xi(\lambda, v, h) := \sigma(\lambda, T^2_K [G(\bar{x}, \bar{u}), DG(\bar{x}, \bar{u})(h, v)])$$

Pseudo Quadratic problem: dual formulation

$$\max_{\lambda \in S(DL)} D^2_{xx} L(\bar{x}, \bar{u}, \lambda)(h, v)^2 - \Xi(\lambda, v, h) \quad (QD_h)$$
Upper estimates: second-order

If (RQC):

$$\text{val}(QP_h) = \text{val}(QD_h), \quad \text{for all } h \in S(LP).$$

**Second-order Upper Estimates:**

$$\text{val}(u_\tau) \leq \text{val}(\bar{u}) + \tau \text{val}(LP) + \frac{1}{2} \tau^2 \text{val}(QD_h) + o(\tau^2).$$

Best estimate:

$$\min_{h \in S(LP)} \text{val}(QD_h). \quad (QD)$$

$$\text{val}(u_\tau) \leq \text{val}(\bar{u}) + \tau \text{val}(LP) + \frac{1}{2} \tau^2 \text{val}(QD) + o(\tau^2).$$
Lower estimates: first-order

Let $x_\tau$ be a $o(\tau)$ solution of $(P_{u_\tau})$:
Use the previous first-order upper estimate and $\lambda \in N_K[G(\bar{x}, \bar{u})]$: 

$$f(x_\tau, u_\tau) - f(\bar{x}, \bar{u}) = \text{val}(u_\tau) - \text{val}(\bar{u}) + o(\tau) \leq \tau \text{val}(LD) + o(\tau)$$

$$\langle \lambda, G(x_\tau, u_\tau) - G(\bar{x}, \bar{u}) \rangle \leq 0.$$ 

Sum of above inequalities:

$$L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) \leq \text{val}(u_\tau) - \text{val}(\bar{u}) \leq \tau \text{val}(LD) + o(\tau)$$

Take $\lambda \in S(DL)$: then $\text{val}(L) = D_u L(\bar{x}, \bar{u}, \lambda)$ and so

$$L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) = \tau \text{val}(LD) + O(\tau^2 + \|x_\tau - \bar{x}\|^2)$$

If (stability result) $\|x_\tau - \bar{x}\| = o(\tau^{1/2})$ deduce the marginal cost

(in direction $v$)

$$\text{val}(u_\tau) = \text{val}(\bar{u}) + \tau \text{val}(LD) + o(\tau).$$
Lower estimates: second-order I

Assume existence of $x_\tau$, $o(\tau^2)$ solution of $(P_{u_\tau})$, with

$$\|x_\tau - \bar{x}\| = O(\tau)$$

$$L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) \leq f(x_\tau, u_\tau) - \text{val}(\bar{u}) \leq \tau \text{val}(LD) + \frac{1}{2} \tau^2 \text{val}(QD) + o(\tau^2).$$

For $\lambda \in S(DL)$, the l.h.s. is equal to

$$\tau \text{val}(LD) + \frac{1}{2} D^2_{(x,u)} L(\bar{x}, \bar{u}, \lambda)(x_\tau - \bar{x}, \tau v)^2 + o(\tau^2).$$

Cancelling first-order terms and setting $h_\tau := (x_\tau - \bar{x})/\tau$, get:

$$\max_{\lambda \in S(DL)} D^2_{(x,u)} L(\bar{x}, \bar{u}, \lambda)(h_\tau, v)^2 \leq \text{val}(QD) + o(1)$$
Passing to the limit in the solution

- If $h_\tau \rightharpoonup \bar{h}$ (weak convergence, for an extracted sequence):
  $\bar{h} \in S(LP)$.

- **Legendre** positively homogeneous form $Q : X \to R$: w.l.s.c.
  and
  
  $$h_\tau \rightharpoonup \bar{h} \text{ and } Q(h_\tau) \rightharpoonup Q(\bar{h}) \text{ implies } h_\tau \rightarrow \bar{h}.$$  

- If $h \mapsto \max_\lambda \in S(LD) D_{xx}^2 L(\bar{x}, \bar{u}, \lambda)(h, h)$ is Legendre then by previous expansions and $\Xi(\lambda, v, h) \leq 0$:

  $$\text{val}(QD_h) = \max_{\lambda \in S(LD)} D_{xx}^2 L(\bar{x}, \bar{u}, \lambda)(\bar{h}, \bar{h}) \leq \text{val}(QD)$$  

  But if $\Xi(\lambda, v, \bar{h}) = 0$ this implies $h \in S(QD)$!

  (last hypothesis often obtained through reduction theory)
General framework
Upper estimates of the value function
State constrained optimal control problems
Open problems

Lower estimates

Lower estimates: second-order II

Theorem

Assume (i) \((RQC)\), (ii) existence of \(x_\tau\), \(o(\tau^2)\) solution of \((P_{u_\tau})\), such that \(\|x_\tau - \bar{x}\| = O(\tau)\), (iii) \(\text{val}(QD)\) finite, (iv) \(h \mapsto \max_{\lambda \in S(LD)} D^2_{xx} L(\bar{x}, \bar{u}, \lambda)(h, h)\) Legendre, (v) \(\Xi(\lambda, \nu, \bar{h}) = 0\)

Then

\[
\text{val}(u_\tau) = \text{val}(\bar{u}) + \tau \text{val}(LP) + \frac{1}{2} \tau^2 \text{val}(QD) + o(\tau^2).
\]

In in addition \((QD)\) has a unique solution \(\hat{h}\), then \(\bar{h} = \hat{h}\) and

\[
x_\tau = \bar{x} + \tau \bar{h} + o(\tau).
\]

Ref FB-Shapiro book (2000) + its refs
Optimal control with state constraints

- **State equation:** \( y(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \)
  \[
  \dot{y}(t) = f(u(t), y(t)) \quad \text{p.p. } t \in [0, T], \quad y(0) = y_0
  \] (1)

- **State constraint:**
  \[
  g_i(y(t)) \leq 0, \quad t \in [0, T], \quad i = 1, \ldots, r.
  \] (2)

- **Same cost function:** integral + final term
  \[
  J(u, y) = \int_0^T \ell(u(t), y(t))\,dt + \phi(y(T)).
  \] (3)

- **Optimal control problem**
  \[
  \min_{(u, y)} J(u, y) \quad \text{s.t. (1) and (2).}
  \] (P)

- \( C^\infty, \) **Lipschitz data** \( f, \ell, \phi, g. \)
Order of the state constraint

- **Total derivative of a scalar state constraint:**

  \[ g^{(1)}(u, y) := g'(y)f(u, y). \]

  While result does not depend on \( u \), we can continue:

  \[ g^{(i+1)}(u, y) := g^{(i)}(y)f(u, y). \]

  **Constraint order:** \( q \) smallest number such that

  \[ g_u^{(q)}(u, y) \neq 0 \]

  **Well-posed constraint order:** when

  \[ g_u^{(q)}(u, y) \neq 0, \quad \text{for all } (u, y) \]
**Constraint structure**

- **Contact set:** \( \{ t \in [0, T] ; g(y(t)) = 0 \} \).

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g(y(t))
```

- **Boundary arc** \([\tau_{en}, \tau_{ex}]\)  
- **(isolated) touch point** \(\{\tau_{to}\}\)

**Question:** If known structure: number, ordering of boundary arcs and touch points;  
Then can we design a shooting algorithm?  
Will it be well-posed?
Junction points

- **Set of junction points**: closure of end-points of interior arcs
- **Regular junction point**: end-point of two arcs, of three types:
  - **Entry, exit points**: end-points of a boundary arc
  - **Touch point**: isolated contact points
(P^\mu) \min_{(u,y) \in U \times Y} \int_0^T \ell^\mu(u(t), y(t)) dt + \phi^\mu(y(T))

s.c. \quad \dot{y}(t) = f^\mu(u(t), y(t)) \text{ p.p. } [0, T] \ ; \ y(0) = y_0^\mu

\quad g^\mu(y(t)) \leq 0 \quad \text{on } [0, T].

- Rem. : scalar control \( u(t) \in \mathbb{R}. \)
- \( \mu \) : perturbation parameter
- Hyp (A0) smooth data: \( C^\infty, \) Lipschitz (A1) \( g^{\mu_0}(y_0^{\mu_0}) < 0. \)
Hypotheses I

\[(\tilde{u}, \tilde{y})\] solution for \(\mu = \mu_0\), with multipliers \((\tilde{p}, \tilde{\eta})\).

(A2) \(H^{\mu_0}(\cdot, \tilde{y}(t), \tilde{p}(t))\) uniformly strongly convex where

\[H(u, y, p, \mu) := \ell^\mu(u, y) + pf^\mu(u, y).\]

(A3) (Order 1 constraint) for all \(t\):

\[|g_u^{(1)}(u(t), y(t))| \geq \gamma > 0.\]
Hypotheses II

(A4) \((\bar{u}, \bar{y})\) has a finite number of regular junctions.

(A5) Strict complementarity on boundary arcs:

\[
\frac{d\bar{\eta}(t)}{dt} \geq \beta > 0, \quad \text{on interior boundary arcs.}
\]

(A6) For all touch point (isolated contact point) \(\tau\),

\[
\frac{d^2}{dt^2} g(\bar{y}(t))|_{t=\tau} < 0.
\]
Notion of quadratic growth condition

**Def:** The Quadratic Growth Condition (QGC) holds if, for all $C^2$-perturbation $(P^\mu)$ of $(P^{\mu_0})$, there exists a neighborhood $(V_u, V_\mu)$ of $(\bar{u}, \mu_0)$, and constants $c, r > 0$ such that for $\mu \in V_\mu$, there exists a unique local solution $(u^\mu, y^\mu)$ of $(P^\mu)$ with $u^\mu \in V_u$ satisfying

\[
J^\mu(u, y) \geq J^\mu(u^\mu, y^\mu) + c\|u - u^\mu\|_2^2,
\]

\forall (u, y) \text{ feasible for } (P^\mu), \|u - \bar{u}\|_\infty < r.
Main result: statement

\textbf{Theorem}

Let \((\bar{u}, \bar{y}) = (u_{\mu_0}, y_{\mu_0})\) local solution of \((P_{\mu_0})\) satisfying (A1)-(A6). Then the following statements are equivalent:

(i) The QGC holds

(ii) The following second-order sufficient condition is satisfied: The tangent linear-quadratic problem (defined later) has \(v = 0\) as unique solution.

Under these conditions: local uniqueness of local solutions in \(\mathcal{U}\).
Also: Boundary arcs are stable,
Touch point remain so, vanish or become boundary arcs.
... If (i) or (ii) is satisfied, then \( \mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu) \) is locally Lipschitz in

\[ U \times Y \times L^\infty(0, T; \mathbb{R}^{n*}) \times L^\infty(0, T; \mathbb{R}) \]

and directionally differentiable in

\[ L^r(0, T) \times W^{1,r}(0, T; \mathbb{R}^n) \times L^r(0, T; \mathbb{R}^{n*}) \times L^r(0, T) \]

for all \( 1 \leq r < \infty \). The directional derivative in direction \( d \) is the unique solution of a certain linear quadratic problem \( (P_d) \).
The linear quadratic problem

Space of linearized control and states

\[ \mathcal{V} := L^2(0, T) \supset \mathcal{U}; \quad \mathcal{Z} := H^1(0, T; \mathbb{R}^n) \supset \mathcal{Y}. \]

\[ d = \mu - \mu_0 : \text{“given” direction of perturbation.} \]

\[ (\mathcal{P}_d) \quad \min_{(v, z) \in \mathcal{V} \times \mathcal{Z}} \frac{1}{2} \int_0^T D^2_{(u, y, \mu)} H^{\mu_0}(\bar{u}, \bar{y}, \bar{p})(v, z, d)^2 \, dt \]

\[ + D^2 \phi^{\mu_0}(\bar{y}(T))(z(T), d)^2 + \int_0^T D^2 g^{\mu_0}(\bar{y})(z, d)^2 \, d\tilde{\eta}(t) \]

s.c. \[ \dot{z}(t) = Df^{\mu_0}(\bar{u}, \bar{y})(v, z, d) \quad \text{sur } [0, T], \quad z(0) = Dy_0^{\mu_0} d \]

\[ Dg^{\mu_0}(\bar{y})(z, d) = 0 \quad \text{on boundary arcs of } (\bar{u}, \bar{y}) \]

\[ Dg^{\mu_0}(\bar{y}(\tau))(z(\tau), d) \leq 0, \quad \forall \tau \text{ isolated contact point of } (\bar{u}, \bar{y}). \]
Algorithmic consequences

- If no isolated touch point: Newton’s method well-defined (with the “shooting parameters, see paper)
- Convergent homotopy algorithm taking into account transitions
- Touch point viewed as zero length boundary arc
- Backtracking over $\mu$ if Newton’s method non convergent.
Expression of linearization of entry times

Linearize

$$\hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0) = 0$$

Denote by $v$, $z$, $\sigma^{en}$ the directional derivative of control, state, entry point w.r.t. a variation of $\mu$ in direction $d$, then

$$D \hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0)(v(\bar{t}^{en-}), z(\bar{t}^{en}), d) + \sigma^{en} \frac{d}{dt} g^{(1)}(\bar{u}, \bar{y})|_{t=\bar{t}^{en-}} + 0,$$

from which we can extract $\sigma^{en}$:

$$\sigma^{en} = - \frac{D \hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0)(v(\bar{t}^{en-}), z(\bar{t}^{en}), d)}{\frac{d}{dt} g^{(1)}(\bar{u}, \bar{y})|_{t=\bar{t}^{en-}}}$$
Open problems

- Optimal control
- Variational problems
- Stochastic control
Initial-final state constraints, in the case of “bounded strong” minima, with nonunique multipliers:

\[ \|u\|_\infty \leq M \text{ and } \|y - \bar{y}\|_\infty \leq \varepsilon. \]

Same problem with additional control constraints.
Idem, with distributed state constraints.
Problems with control entering linearly
Logarithmic penalty

See work by Milyutin, Osmolovskii, Dmitruk, Goh.
Perturbation of a static plate problem with obstacle:

$$\text{Min} \quad \frac{1}{2} \int_{\Omega} (\Delta u(x))^2 dx - \int_{\Omega} f(x)u(x)dx; \quad u(x) \geq \Psi(x)$$

with $\Omega$ bounded smooth open in $\mathbb{R}^2$, $u = \frac{\partial u}{\partial n}$ over $\partial \Omega$.

see C. Pozzolini’s thesis; Rao and Sokolowski, Mignot, Haraux.
Stochastic control

State equation

\[ dy_t = f(u_t, y_t)dt + \sigma(u_t, y_t)dW_t, \quad y_0 = y^0 \]

Cost function

\[ \int_0^T \ell(u_t, y_t)dt + \phi(y_T) \]

\( W_t \) standard Brownian, all functions Lipschitz and bounded

Control adapted to the Brownian filtration

Expansion of optimal control (see Bensoussan’s book (1988) in unconstrained case)
References