Sensitivity analysis for optimization and optimal control problems

J. Frédéric Bonnans

INRIA-Saclay & CMAP, Ecole Polytechnique, France

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Families of optimization problems

Generic framework:

$$\operatorname*{Min}_{x} \; f(x,u); \quad G(x,u) \in K \qquad (P_u)$$

with

- U, X, Y: Banach spaces
- K closed convex subset of Y
- $f : X \times U \to \mathbb{R}$ and $G : X \times U \to Y$, smooth (C^∞).

Examples

- $K = \{0\}_{\mathbb{R}^p} \times \mathbb{R}^q_-$: finitely may equalities and inequalities. If $X = \mathbb{R}^n$: **Nonlinear programming**.
- **Semi definite programming**: $X = \mathbb{R}^n$ and $K = \mathcal{S}^p_-$, with $\mathcal{S}^p$ set of symmetric matrices of order p, or more generally

$$K = \mathcal{S}^{p_1}_- \times \cdots \mathcal{S}^{p_q}_-$$

- **Semi infinite programming**: $X = \mathbb{R}^n$, Ω compact metric, $Y = C(\Omega)$, $K = C(\Omega)_-$:
- **Optimal control of ODEs**
- **Optimal control of PDEs**

Notations

$$\operatorname*{Min}_{x} f(x,u); \quad G(x,u) \in K \qquad (P_u)$$

- Set of feasible points: $F(u) = G^{-1}(\cdot, u)(K)$. Value

$$\mathrm{val}(u) := \inf\{f(x,u);\ x \in F(u)\}$$

Set of solutions

$$S(u) := \{x \in F(u);\ f(x,u) = \mathrm{val}(u)\}.$$

Questions

- Stability of solutions:

$$\operatorname{dist}[S(u'), S(u)] = O(\|u' - u\|) \text{ or } O(\|u' - u\|^{1/2})$$

- Expansion of (approximate) solutions along a path $u_\tau := \bar{u} + \tau v$, $\tau \in \mathbb{R}_+$

$$S(u_\tau) = \bar{x} + \tau^\gamma h + o(\tau^\gamma), \quad \gamma = 1 \text{ or } 1/2$$

- Expansion of value function: find subproblems (L) and (Q) such that

$$\operatorname{val}(u_\tau) = \operatorname{val}(\bar{u}) + \tau \operatorname{val}(L) + \tfrac{1}{2}\tau^2 \operatorname{val}(Q) + o(\tau^2).$$

First and second order tangent cones

Nominal problem: $f(x) = f(x, \bar{u})$, $G(x) = G(x, \bar{u})$.

Here $\mathcal{K} \subset X$, possibly nonconvex, e.g. $\mathcal{K} = G^{-1}(K)$:

- **(Interior) tangent cone**: $y \in \mathcal{K}$

$$T_{\mathcal{K}}(x) = \{y \in X;\ \mathrm{dist}(x + \tau y, \mathcal{K}) = o(\tau),\ \tau \geq 0\}$$

- **(Interior) second order tangent set**: $x \in \mathcal{K}$, $y \in T_{\mathcal{K}}(x)$.

$$T^2_{\mathcal{K}}(x, y) = \{z \in Y;\ \mathrm{dist}(x + \tau y + \tfrac{1}{2}\tau^2 z, \mathcal{K}) = o(\tau^2),\ \tau \geq 0\}$$

Second-order necessary condition

Nominal problem again:

$$\operatorname*{Min}_x f(x); \quad x \in \mathcal{K} \subset X.$$

Use

$$\begin{aligned} f(x+\tau y+\tfrac{1}{2}\tau^2 z) = \quad & f(x)+\tau Df(x)y \\ & +\tfrac{1}{2}\tau^2[Df(x)z+D^2f(x)(y,y)]+o(\tau^2) \end{aligned}$$

First-order necessary condition:

$$Df(\bar{x})h \geq 0, \quad \text{for all } \; h \in T_{\mathcal{K}}(x)$$

Second-order necessary condition: for all
$h \in C(\bar{x}) := T_{\mathcal{K}}(x) \cap Df(\bar{x})^{\perp}$:

$$Df(\bar{x})z + D^2 f(\bar{x})(h,h) \geq 0, \quad \text{for all } \; z \in T^2_{\mathcal{K}}(x,h)$$

Robinson's Qualification Condition

Robinson's qualification condition (RQC) for $G(x) \in K$:

$$\text{For some } \varepsilon > 0, \quad \varepsilon B \subset G(x) + \mathrm{Im}[DG(x)] - K \qquad (RQC)$$

Specific cases:

- Equality constraints: $K = \{0\}$, reduces to $DG(x)$ onto
- $\mathrm{int}(K) \neq \emptyset$, reduces to

$$\text{For some } h \in X, \quad G(x) + DG(x)h \in \mathrm{int}(K)$$

Normal cone

$$N_K(y) = \{y^* \in Y^*;\ \langle y^*, y' - y\rangle \leq 0, \ \text{for all } y' \in K\}$$

Calculus of first and second order tangent cones

If (RQC) holds then we have

$$\begin{aligned} G(x+\tau y+\tfrac{1}{2}\tau^2 z) = \; & G(x)+\tau DG(x)y \\ & +\tfrac{1}{2}\tau^2[DG(x)z+D^2G(x)(z,z)]+o(\tau^2) \end{aligned}$$

- **Tangent cone**: $y \in \mathcal{K}$

$$T_{\mathcal{K}}(x) = \{y \in Y;\; DG(x)y \in T_K(G(x))\}$$

- **Second order tangent set**: $x \in \mathcal{K}$, $y \in T_{\mathcal{K}}(x)$.

$$\begin{aligned} T^2_{\mathcal{K}}(x,y) = \{w \in Y^*; DG(x)z + D^2G(x)(y,y) \\ \in T^2_K[G(x), DG(x)y]\} \end{aligned}$$

First-order necessary conditions

If (RQC) holds; **Tangent directions**

$$T_{\mathcal{K}}(\bar{x}) = \{h \in X; DG(\bar{x})h \in T_K(G(\bar{x}))\}$$

If (RQC) holds; **Critical cone**

$$C(\bar{x}) = \{h \perp Df(\bar{x}); \quad DG(\bar{x})h \in T_K(G(\bar{x}))\}$$

Lagrangian function: $L(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle$.
Set of Lagrange multipliers non empty and bounded at local solutions

$$\Lambda(\bar{x}) := \{\lambda \in N_K(G(\bar{x})); \quad D_x L(\bar{x}, \lambda) = 0\}.$$

Robinson 76, Zowe and S. Kurcyusz 79

second-order necessary conditions II

(Cominetti 90) **Primal form**:
For all $h \in C(\bar{x})$, $\mathrm{val}(Q_h) \geq 0$, where

$$\begin{array}{ll} Df(\bar{x})z + D^2 f(\bar{x})(h,h) & \geq 0 \text{ if} \\ DG(\bar{x})z + D^2 G(\bar{x})(h,h) & \in T_K^2[G(x), DG(x)h] \end{array} \qquad (Q_h)$$

Support function: $\sigma(\lambda, K) := \sup\{\langle \lambda, w \rangle;\ w \in K\}$
Support function of second-order tangent set (Curvature term)

$$\Xi(\lambda, h) := \sigma(\lambda, T_K^2[G(\bar{x}), DG(\bar{x})h])$$

Lagrangian of (Q_h):

$$D_x L(x,\lambda)z + D_x^2 L(x,\lambda)(h,h) - \Xi(\lambda, h).$$

Dual form of second-order necessary conditions:

$$\operatorname*{Max}_{\lambda \in \Lambda(\bar{x})} D_{xx}^2 L(x,\lambda)(z,z) - \Xi(\lambda, h) \geq 0.$$

Sign of $\sigma(\lambda, T_K^2(g, \delta g))$

When $\lambda \in N_K(G(\bar{x}))$, $\lambda \perp \delta g$ and $w \in T_K^2(g, \delta g)$:

$$0 \geq \langle \lambda, g + \tau\delta g + \tfrac{1}{2}\tau^2 w + o(\tau^2)\rangle = \tau^2 \left[\langle \lambda, w\rangle + o(1)\right]$$

and so whe always have $\Xi(\lambda, h) \leq 0$. In particular, the second-order necessary condition is weaker than the condition

$$\operatorname*{Max}_{\lambda \in \Lambda(\bar{x})} D^2_{xx} L(x, \lambda)(z, z) \geq 0.$$

Cases when $\Xi(\lambda, h) = 0$:
(a) K polyhedron
(b) K pointed cone and $G(\bar{x}) = 0$.
Reduction theory: rewriting of (P) such that (b) holds
(FB and Shapiro 2000)

Example: inequality constraints I

For $K = \mathbb{R}^q_-$, assuming (RCP) $\equiv$ Mangasarian-Fromovitz
Define sets of active constraints

$$\begin{aligned} I(x) &:= \{1 \leq i \leq q;\ G_i(x) = 0\} \\ I(x,y) &:= \{i \in I(x);\ DG_i(x)y = 0\} \end{aligned}$$

- **Tangent cone**: $x \in \mathcal{K}$,

$$T_{\mathcal{K}}(x) = \{y \in X; DG_i(x)z \leq 0, i \in I(x)\}$$

Set of Lagrange multipliers

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^p_+;\ \lambda \cdot G(x) = 0;\ Df(\bar{x}) + DG(x)^\top \lambda = 0\}$$

Example: inequality constraints II

- **Second order tangent cone**: $x \in \mathcal{K}$, $y \in T_{\mathcal{K}}(y)$.

$$T^2_{\mathcal{K}}(x,y) = \{z \in X; DG_i(x)z + D^2G_i(x)(y,y) \leq 0, i \in I(x,y)\}$$

Second order necessary conditions

$$\max_{\lambda \in \Lambda(\bar{x})} D^2_{xx} L(\bar{x}, \lambda)(h,h) \geq 0.$$

Example: semidefinite programming I

For $K = \mathcal{S}^p_-$, we have

- Qualification: $G(x) + DG(x)\hat{y} \in \mathcal{S}_{--}$ (negative definite) for some $\hat{y}$
- E $p \times s$ matrix whose columns are an orthonormal basis of the kernel of $A = G(x)$. **Tangent cone**:

$$T_{\mathcal{S}^p_-}(A) = \{H \in \mathcal{S}^p; E^\top HE \preceq 0\}$$

- F $s \times k$ matrix whose columns are an orthonormal basis of the eigenspace associated with the largest eigenvalue of $E^\top HE$. **Second order tangent cone**: $x \in \mathcal{K}$, $y \in T_{\mathcal{K}}(y)$.

$$T^2_{\mathcal{S}^p_-}(A, H) = \{W \in \mathcal{S}^p; F^\top E^\top WEF \preceq 2F^\top E^\top HA^\dagger HEF\}$$

with $A^\dagger$ pseudo-inverse of A.

Example: semidefinite programming II

Support function of second-order tangent set: quadratic function

$$\Xi(\lambda, h) = -h^\top \mathcal{H}(\bar{x}, \lambda) h$$

with, setting $G_i(\bar{x}) = \partial G(\bar{x})/\partial x_i$:

$$[\mathcal{H}(\bar{x}, \lambda)]_{ij} = -2\lambda \circ \left(G_i(\bar{x})[G(\bar{x})^\dagger] G_j(\bar{x}) \right)$$

Ref. A. Shapiro, 1997.

Upper estimates: first-order

Generic framework: path $u_\tau := \bar{u} + \tau v$

$$\operatorname*{Min}_{x} \; f(x,u); \quad G(x,u) \in K \tag{P_u}$$

$\bar{x} \in S(\bar{u})$; **Linearized problem**

$$\begin{array}{ll} \mathrm{Min}_{h \in X} & Df(\bar{x},\bar{u})(h,v); \\ & DG(\bar{x},\bar{u})(h,v) \in T_K[G(\bar{x},\bar{u})]. \end{array} \tag{LP}$$

Lagrangian function: $L(x,u,\lambda) := f(x,u) + \langle \lambda, G(x,u) \rangle$.
If (RQC): $\mathrm{val}(LP) = \mathrm{val}(LD)$ where

$$\operatorname*{Max}_{\lambda \in \Lambda(\bar{x})} D_u L(\bar{x},\bar{u},\lambda)v \tag{LD}$$

Set of dual solutions: $S(LD)$. **First-order upper estimate**

$$\mathrm{val}(u_\tau) \leq \mathrm{val}(\bar{u}) + \tau\, \mathrm{val}(LP) + o(\tau).$$

Upper estimates: second-order, pseudo quadratic problem

Pseudo Quadratic problem: primal formulation

$\mathrm{val}(QP_h) \geq 0$, where

$$\begin{array}{ll} \mathrm{Min}_{z\in X} & D_x f(\bar{x},\bar{u})(z,z) + D^2 f(\bar{x},\bar{u})(h,v)^2 \\ & D_x G(\bar{x},\bar{u})z + D^2 G(\bar{x},\bar{u})(h,v)^2 \\ & \in T^2_K[G(\bar{x},\bar{u}), DG(\bar{x},\bar{u})(h,v)]. \end{array} \qquad (QP_h)$$

Curvature term for the perturbation problem

$$\Xi(\lambda, v, h) := \sigma(\lambda, T^2_K[G(\bar{x},\bar{u}), DG(\bar{x},\bar{u})(h,v)])$$

Pseudo Quadratic problem: dual formulation

$$\underset{\lambda\in S(DL)}{\mathrm{Max}} \; D^2_{xx} L(\bar{x},\bar{u},\lambda)(h,v)^2 - \Xi(\lambda, v, h) \qquad (QD_h)$$

Upper estimates: second-order

If (RQC):

$$\mathrm{val}(QP_h) = \mathrm{val}(QD_h), \quad \text{for all } h \in S(LP).$$

Second-order Upper Estimates:

$$\mathrm{val}(u_\tau) \leq \mathrm{val}(\bar{u}) + \tau\,\mathrm{val}(LP) + \tfrac{1}{2}\tau^2\,\mathrm{val}(QD_h) + o(\tau^2).$$

Best estimate:

$$\operatorname*{Min}_{h \in S(LP)} \mathrm{val}(QD_h). \qquad (QD)$$

$$\mathrm{val}(u_\tau) \leq \mathrm{val}(\bar{u}) + \tau\,\mathrm{val}(LP) + \tfrac{1}{2}\tau^2\,\mathrm{val}(QD) + o(\tau^2).$$

Lower estimates: first-order

Let x_τ be a $o(\tau)$ solution of (P_{u_τ}):
Use the previous first-order upper estimate and $\lambda \in N_K[G(\bar{x}, \bar{u})]$:

$$\begin{aligned} f(x_\tau, u_\tau) - f(\bar{x}, \bar{u}) = \mathrm{val}(u_\tau) - \mathrm{val}(\bar{u}) + o(\tau) &\leq \tau\, \mathrm{val}(LD) + o(\tau) \\ \langle \lambda, G(x_\tau, u_\tau) - G(\bar{x}, \bar{u}) \rangle &\leq 0. \end{aligned}$$

Sum of above inequalities:

$$L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) \leq \mathrm{val}(u_\tau) - \mathrm{val}(\bar{u}) \leq \tau\, \mathrm{val}(LD) + o(\tau)$$

Take $\lambda \in S(DL)$: then $\mathrm{val}(L) = D_u L(\bar{x}, \bar{u}, \lambda)$ and so

$$L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) = \tau\, \mathrm{val}(LD) + O(\tau^2 + \|x_\tau - \bar{x}\|^2)$$

If (stability result) $\|x_\tau - \bar{x}\| = o(\tau^{1/2})$ deduce the **marginal cost** (in direction v)

$$\mathrm{val}(u_\tau) = \mathrm{val}(\bar{u}) + \tau\, \mathrm{val}(LD) + o(\tau).$$

Lower estimates: second-order I

Assume existence of x_τ, $o(\tau^2)$ solution of (P_{u_τ}), with $\|x_\tau - \bar{x}\| = O(\tau)$

$$\begin{aligned} L(x_\tau, u_\tau, \lambda) - L(\bar{x}, \bar{u}, \lambda) &\leq f(x_\tau, u_\tau) - \mathrm{val}(\bar{u}) \\ &\leq \tau \, \mathrm{val}(LD) + \tfrac{1}{2}\tau^2 \, \mathrm{val}(QD) + o(\tau^2). \end{aligned}$$

For $\lambda \in S(DL)$, the l.h.s. is equal to

$$\tau \, \mathrm{val}(LD) + \tfrac{1}{2} D^2_{(x,u)^2} L(\bar{x}, \bar{u}, \lambda)(x_\tau - \bar{x}, \tau v)^2 + o(\tau^2).$$

Cancelling first-order terms and setting $h_\tau := (x_\tau - \bar{x})/\tau$, get:

$$\max_{\lambda \in S(DL)} D^2_{(x,u)^2} L(\bar{x}, \bar{u}, \lambda)(h_\tau, v)^2 \leq \mathrm{val}(QD) + o(1)$$

Passing to the limit in the solution

- If $h_\tau \rightharpoonup \bar{h}$ (weak convergence, for an extracted sequence): $\bar{h} \in S(LP)$.
- **Legendre** positively homogeneous form $Q : X \to \boldsymbol{R}$: w.l.s.c. and
$$h_\tau \rightharpoonup \bar{h} \text{ and } Q(h_\tau) \to Q(\bar{h}) \text{ implies } h_\tau \to \bar{h}.$$
- If $h \mapsto \max_{\lambda \in S(LD)} D^2_{xx} L(\bar{x}, \bar{u}, \lambda)(h, h)$ is Legendre then by previous expansions and $\Xi(\lambda, v, h) \leq 0$:
$$\mathrm{val}(QD_h) = \max_{\lambda \in S(LD)} D^2_{xx} L(\bar{x}, \bar{u}, \lambda)(\bar{h}, \bar{h}) \leq \mathrm{val}(QD)$$
But if $\Xi(\lambda, v, \bar{h}) = 0$ this implies $h \in S(QD)$!
(last hypothesis often obtained through reduction theory)

Lower estimates: second-order II

Theorem

Assume (i) (RQC), (ii) existence of x_τ, $o(\tau^2)$ solution of (P_{u_τ}), such that $\|x_\tau - \bar{x}\| = O(\tau)$, (iii) $\mathrm{val}(QD)$ finite, (iv) $h \mapsto \max_{\lambda \in S(LD)} D^2_{xx} L(\bar{x}, \bar{u}, \lambda)(h,h)$ Legendre, (v) $\Xi(\lambda, v, \bar{h}) = 0$
Then

$$\mathrm{val}(u_\tau) = \mathrm{val}(\bar{u}) + \tau\, \mathrm{val}(LP) + \tfrac{1}{2}\tau^2\, \mathrm{val}(QD) + o(\tau^2).$$

In in addition (QD) has a unique solution $\hat{h}$, then $\bar{h} = \hat{h}$ and $x_\tau = \bar{x} + \tau \bar{h} + o(\tau)$.

Ref FB-Shapiro book (2000) + its refs

Optimal control with state constraints

- State equation: $y(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

$$\dot{y}(t) = f(u(t), y(t)) \qquad \text{p.p. } t \in [0, T], \qquad y(0) = y_0 \quad (1)$$

- State constraint:

$$g_i(y(t)) \leq 0, \quad t \in [0, T],\ i = 1, \dots, r. \qquad (2)$$

- Same cost function: integral + final term

$$J(u, y) = \int_0^T \ell(u(t), y(t))\mathrm{d}t + \phi(y(T)). \qquad (3)$$

- Optimal control problem

$$\operatorname*{Min}_{(u,y)} J(u, y) \quad \text{s.t. (1) and (2).} \qquad (P)$$

- C^∞, Lipschitz data f, ℓ, ϕ, g.

Order of the state constraint

- Total derivative of a scalar state constraint:

$$g^{(1)}(u,y) := g'(y)f(u,y).$$

While result does not depend on u, we can continue:

$$g^{(i+1)}(u,y) := g^{(i)}(y)f(u,y).$$

Constraint order: q smallest number such that

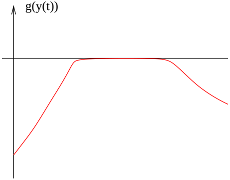
$$g_u^{(q)}(u,y) \neq 0$$

Well-posed constraint order: when

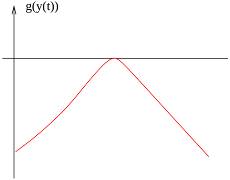
$$g_u^{(q)}(u,y) \neq 0, \quad \text{for all } (u,y)$$

Constraint structure

- Contact set: $\{t \in [0, T] \; ; \; g(y(t)) = 0\}$.



boundary arc $[\tau_{en}, \tau_{ex}]$



(isolated) touch point $\{\tau_{to}\}$

- Question: If known structure: number, ordering of boundary arcs and touch points;
  Then can we design a shooting algorithm ?
  Will it be well-posed ?

Junction points

- Set of junction points: closure of end-points of interior arcs
- Regular junction point: end-point of two arcs, of three types:
- Entry, exit points: end-points of a boundary arc
- Touch point: isolated contact points

Sensitivity: Framework

$$(P^\mu) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell^\mu(u(t),y(t))\mathrm{d}t + \phi^\mu(y(T))$$

$$\text{s.c.} \qquad \dot{y}(t) = f^\mu(u(t),y(t)) \text{ p.p. } [0,T]\ ;\ y(0) = y_0^\mu$$

$$g^\mu(y(t)) \leq 0 \quad \text{on } [0,T].$$

- Rem. : scalar control $u(t) \in \mathbb{R}$.
- μ : perturbation parameter
- Hyp **(A0)** smooth data: C^∞, Lipschitz **(A1)** $g^{\mu_0}(y_0^{\mu_0}) < 0$.

Hypotheses I

$(\bar{u}, \bar{y})$ solution for $\mu = \mu_0$, with multipliers $(\bar{p}, \bar{\eta})$.

(A2) $H^{\mu_0}(\cdot, \bar{y}(t), \bar{p}(t))$ uniformly strongly convex where

$$H(u, y, p, \mu) := \ell^{\mu}(u, y) + pf^{\mu}(u, y).$$

(A3) (Order 1 constraint) for all t:

$$|g_u^{(1)}(u(t), y(t))| \geq \gamma > 0.$$

Hypotheses II

(A4) $(\bar{u}, \bar{y})$ has a finite number of regular junctions.

(A5) Strict complementarity on boundary arcs:

$$\frac{\mathrm{d}\bar{\eta}(t)}{\mathrm{d}t} \geq \beta > 0, \qquad \text{on interior boundary arcs.}$$

(A6) For all touch point (isolated contact point) τ,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\bar{y}(t))|_{t=\tau} < 0.$$

Notion of quadratic growth condition

Def: The Quadratic Growth Condition (QGC) holds) holds if, for all C^2-perturbation (P^{μ}) of (P^{μ_0}), there exists a neighborhood (V_u, V_μ) of $(\bar{u}, \mu_0)$, and constants $c, r > 0$ such that for $\mu \in V_\mu$, there exists a unique local solution (u^μ, y^μ) of $(\mathcal{P}^\mu)$ with $u^\mu \in V_u$ satisfying

$$J^\mu(u, y) \geq J^\mu(u^\mu, y^\mu) + c\|u - u^\mu\|_2^2,$$

$$\forall\, (u, y) \text{ feasible for } (P^\mu),\ \|u - \bar{u}\|_\infty < r.$$

Main result: statement

Theorem

Let $(\bar{u}, \bar{y}) = (u^{\mu_0}, y^{\mu_0})$ local solution of (P^{μ_0}) satisfying (A1)-(A6). Then the following statements are equivalent:

(i) *The QGC holds*

(ii) *The following second-order sufficient condition is satisfied: The tangent linear-quadratic problem (defined later) has $v = 0$ as unique solution.*

Under these conditions: local uniqueness of local solutions in $\mathcal{U}$.

Also: Boundary arcs are stable,

Touch point remain so, vanish or become boundary arcs.

Main result (continued)

Theorem (End of statement)

... If (i) or (ii) is satisfied, then $\mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu)$ is locally Lipschitz in

$$\mathcal{U} \times \mathcal{Y} \times L^\infty(0,T;\mathbb{R}^{n*}) \times L^\infty(0,T;\mathbb{R})$$

and directionally differentiable in

$$L^r(0,T) \times W^{1,r}(0,T;\mathbb{R}^n) \times L^r(0,T;\mathbb{R}^{n*}) \times L^r(0,T)$$

for all $1 \leq r < \infty$. The directional derivative in direction d is the unique solution of a certain linear quadratic problem (P_d).

The linear quadratic problem

Space of linearized control and states

$$\mathcal{V} := L^2(0,T) \supset \mathcal{U}; \qquad \mathcal{Z} := H^1(0,T;\mathbb{R}^n) \supset \mathcal{Y}.$$

$d = \mu - \mu_0$: "given" direction of perturbation.

$$(\mathcal{P}_d) \qquad \min_{(v,z)\in\mathcal{V}\times\mathcal{Z}} \quad \frac{1}{2}\int_0^T D^2_{(u,y,\mu)^2} H^{\mu_0}(\bar{u},\bar{y},\bar{p})(v,z,d)^2 \mathrm{d}t$$
$$+ D^2\phi^{\mu_0}(\bar{y}(T))(z(T),d)^2 + \int_0^T D^2 g^{\mu_0}(\bar{y})(z,d)^2 \mathrm{d}\bar{\eta}(t)$$

s.c. $\dot{z}(t) = Df^{\mu_0}(\bar{u},\bar{y})(v,z,d)$ sur $[0,T]$, $z(0) = Dy_0^{\mu_0} d$

$Dg^{\mu_0}(\bar{y})(z,d) = 0$ on boundary arcs of $(\bar{u},\bar{y})$

$Dg^{\mu_0}(\bar{y}(\tau))(z(\tau),d) \leq 0$, $\forall\, \tau$ isolated contact point of $(\bar{u},\bar{y})$.

Algorithmic consequences

- If no isolated touch point: Newton's method well-defined (with the "shooting parameters, see paper)
- Convergent homotopy algorithm taking into account transitions
- Touch point viewed as zero lenght boundary arc
- Backtracking over μ if Newton's method non convergent.

Expression of linearization of entry times

Linearize

$$\hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0) = 0$$

Denote by v, z, σ^{en} the directional derivative of control, state, entry point w.r.t. a variation of μ in direction d, then

$$D\hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0)(v(\bar{t}^{en-}), z(\bar{t}^{en}), d) \\ + \sigma^{en} \frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(\bar{u}, \bar{y})|_{t=\bar{t}^{en-}} + 0,$$

from which we can extract σ^{en}:

$$\sigma^{en} = -\frac{D\hat{g}^{(1)}(\bar{u}(\bar{t}^{en}), \bar{y}(\bar{t}^{en}), \mu_0)(v(\bar{t}^{en-}), z(\bar{t}^{en}), d)}{\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(\bar{u}, \bar{y})|_{t=\bar{t}^{en-}}}$$

Open problems

- Optimal control
- Variational problems
- Stochastic control

Optimal control

- Initial-final state constraints, in the case of “bounded strong” minima, with nonunique multipliers:

$$\|u\|_\infty \leq M \text{ and } \|y - \bar{y}\|_\infty \leq \varepsilon.$$

- Same problem with additional control constraints.
- Idem, with distributed state constraints.
- Problems with control entering linearly
- Logarithmic penalty

See work by Milyutin, Osmolovskiĭ, Dmitruk, Goh.

Variational problems

Perturbation of a static plate problem with obstacle:

$$\operatorname*{Min}_{u} \frac{1}{2} \int_{\Omega} (\Delta u(x))^2 \mathrm{d}x - \int_{\Omega} f(x) u(x) \mathrm{d}x; \quad u(x) \geq \Psi(x)$$

with Ω bounded smooth open in $\mathbb{R}^2$, $u = \partial u / \partial n$ over $\partial\Omega$.

see C. Pozzolini's thesis; Rao and Sokolowski, Mignot, Haraux.

Stochastic control

State equation

$$\mathrm{d}y_t = f(u_t, y_t)\mathrm{d}t + \sigma(u_t, y_t)\mathrm{d}W_t, \quad y_0 = y^0$$

Cost function

$$\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_T)$$

W_t standard Brownian, all functions Lipschitz and bounded
Control adapted to the Brownian filtration

Expansion of optimal control (see Bensoussan's book (1988) in unconstrained case)

References

- Alvarez, J. Bolte, J.F. B., F. Silva: *Asymptotic expansions for interior penalty solutions of control constrained linear-quadratic problems*. INRIA report RR 6863, Mar. 2009.
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