**Mini-course on polynomial optimization and control**
**by Didier Henrion and Jean-Bernard Lasserre**

INSA Rennes, 24-25 March 2014

In collaboration with the conference MODE 2014

# 1 Topic

The mini-course focuses on polynomial optimization and control, with a focus on semidefinite relaxation techniques exploiting duality between moment problems and representations of polynomials nonnegative on semialgebraic sets.

In the early 2000s, several research groups realized that the linear matrix inequality (LMI) framework existing for analysis and control of linear dynamical systems [3] can be extended significantly to nonlinear control systems described by polynomial vector fields. Lyapunov analysis techniques can be extended readily as soon as one notices that positivity constraints on polynomials can be enforced by sum-of-squares (SOS) constraints on polynomials [22]. It turns out that SOS polynomials are semidefinite representable [20], i.e. they can be modelled by projecting a linear section of the semidefinite cone, or cone of nonnegative quadratic forms. In turn, optimization over the semidefinite cone can be efficiently carried out with interior-point methods [1]. Building on results of functional analysis and real algebraic geometry allowing SOS representations of polynomials positive on semialgebraic sets [23], a hierarchy of LMI relaxations was proposed in [15] for polynomial optimization (minimization of a real-valued polynomial subject to finitely many polynomial equalities and inequalities), with convergence guarantees. Of particular importance is the duality between the cone of nonnegative measures (resp. the cone of truncated moment sequences) and the cone of nonnegative functions (resp. the cone of nonnegative polynomials). The cone of truncated moment sequences is approximated from the exterior (relaxed) with appropriate linear sections of the semidefinite cone, whereas the cone of nonnegative polynomials is approximated from the interior (strengthened) with appropriate linear sections of the semidefinite cone.

Public-domain software packages were then released to illustrate the potential of these techniques to solve non-trivial optimization and control problems, see e.g. [7, 8]. Applications in control were surveyed in 2005 in the collective work [9]. The approach was later on extended to polynomial optimal control and infinite-dimensional optimization [16]. After more than a decade, these ideas have been consolidated in a comprehensive survey [19], research monograph [17], a control-oriented survey [4], a collective work reporting on the achievements of a related NSF funded project [2], and more recently, tutorial lecture notes [12].

The objective of this mini-course is to introduce these ideas in a unified, yet accessible fashion, with the hope that it can stimulate further research activities along these lines.

# 2 Lecturers

## 2.1 Didier Henrion

D. Henrion is a CNRS researcher working at LAAS in Toulouse, France. He is also a Professor at the Faculty of Electrical Engineering of the Czech Technical University in Prague, Czech Republic. He is interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory. See `homepages.laas.fr/henrion`

## 2.2 Jean-Bernard Lasserre

J. B. Lasserre is a CNRS researcher working at LAAS and the Institute of Mathematics of the University of Toulouse, France. After contributing to the field of Markov decision problems, he focused on semidefinite programming and linear matrix inequality relaxations for polynomial optimization and optimal control. See `homepages.laas.fr/lasserre`

## 3 Outline

We propose 4 3-hour sessions including a 15-minute break. Each session is directed by a main speaker, but the second speaker may contribute occasionally. Interaction with the audience is expected and encouraged. The first 2 sessions deal with static optimization problems with polynomial objective functions and semialgebraic constraints, whereas the last 2 sessions deal with their dynamic counterparts, or extensions to ordinary differential equations with polynomial vector fields and semialgebraic constraints.

### 3.1 1st session (3 hours) led by Jean-Bernard Lasserre: Semidefinite programming relaxations of nonconvex problems of polynomial optimization

This introductory session focuses on the basics of the hierarchy of LMI relaxations for polynomial optimization [15], focusing on the primal problem of moments and the dual problem of nonnegative polynomials. More recent extensions and developments (rational objective functions, exploiting sparsity structure) are sketched.

### 3.2 2nd session (3 hours) led by Jean-Bernard Lasserre: Representations of nonnegative functions on semialgebraic sets and related problems of moments

This session explains the core mathematical components instrumental to the proof of convergence of the hierarchy of LMI relaxations [23]. Here too the presentation emphasizes the duality relationships between the primal problem of deciding whether a real sequence are moments of a Borel measure, and the dual problem of deciding whether a polynomial is nonnegative on a basic closed semialgebraic set. Extensions to representations of semialgebraic functions are described [18].

### 3.3 3rd session (3 hours) led by Didier Henrion: Numerical methods for designing relaxed control laws in nonlinear optimal control

After explaining the basic ideas [16] allowing to extend the hierarchy of LMI relaxations from static polynomial optimization to dynamic polynomial optimization (that is, infinite-dimensional calculus of variations [5], optimal mass transportation [24], and optimal control [6]), this session describes the software components [11] used to solve numerically, with convergence guarantees, challenging problems of optimal control, especially those with no solutions in classical Lebesgue spaces.

### 3.4 4th session (3 hours) led by Didier Henrion: Numerical methods for computing regions of attraction and maximal controlled invariant sets of nonlinear systems

Finally, this session reports on the most recent extensions of the approach of [16], based on results of [10], allowing the numerical computation of the region of attraction [14] and the maximal controlled invariant set for nonlinear systems [13]. This part puts emphasis on the use of and intuition behind the primal formulation on measures arising in control applications and contrast it with the more traditional dual approach on continuous functions.

## References

[1] A. Ben-Tal, A. Nemirovski. Lectures on modern convex optimization. SIAM, Philadelphia, 2001.

[2] G. Blekerman, P. A. Parrilo, R. R. Thomas (Editors). Semidefinite optimization and convex algebraic geometry. SIAM, Philadelphia, 2013.

[3] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan. Linear matrix inequalities in system and control theory. SIAM, Philadelphia, 1994.

[4] G. Chesi. LMI techniques for optimization over polynomials in control: a survey. IEEE Trans. Autom. Control, 55(11):2500-2510, 2010

[5] B. Dacorogna. Direct methods in the calculus of variations. 2nd edition. Springer-Verlag, Berlin, 2007.

[6] H. O. Fattorini. Infinite dimensional optimization and control theory. Cambridge Univ. Press, Cambridge, UK, 1999.

[7] D. Henrion, J. B. Lasserre. GloptiPoly: global optimization over polynomials with Matlab and SeDuMi. ACM Transactions on Mathematical Software, 29(2):165-194, 2003.

[8] D. Henrion, J. B. Lasserre. Solving nonconvex optimization problems - How GloptiPoly is applied to problems in robust and nonlinear control. IEEE Control Systems Magazine, 24(3):72-83, 2004

[9] D. Henrion, A. Garulli (Editors). Positive polynomials in control. Lecture Notes on Control and Information Sciences, Vol. 312, Springer Verlag, Berlin, 2005.

[10] D. Henrion, J. B. Lasserre, C. Savorgnan. Approximate volume and integration for basic semialgebraic sets. SIAM Review 51(4):722-743, 2009.

[11] D. Henrion, J. B. Lasserre, J. Löfberg. GloptiPoly 3: moments, optimization and semidefinite programming. Optimization Methods and Software, 24(4-5):761-779, 2009.

[12] D. Henrion. Optimization on linear matrix inequalities for polynomial systems control. Les cours du CIRM, Vol.3, No. 1: Journées Nationales de Calcul Formel, 2013.

[13] M. Korda, D. Henrion, C. N. Jones. Convex computation of the maximum controlled invariant set for polynomial control systems, to be presented at the IEEE Conference on Decision and Control, Florence, Italy, 2013.

[14] D. Henrion, M. Korda. Convex computation of the region of attraction of polynomial control systems, to appear in the IEEE Transactions on Automatic Control, 2014.

[15] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11(3):796–817, 2001.

[16] J. B. Lasserre, D. Henrion, C. Prieur, E. Trélat. Nonlinear optimal control via occupation measures and LMI relaxations. SIAM J. Control Optim. 47(4):1643-1666, 2008.

[17] J. B. Lasserre. Moments, positive polynomials and their applications. Imperial College Press, London, UK, 2009.

[18] J. B. Lasserre, M. Putinar. Positivity and optimization for semi-algebraic functions. SIAM J. Optim. 20:3364–3383, 2010.

[19] M. Laurent. Sums of squares, moment matrices and optimization over polynomials. Pages 157-270 in M. Putinar, S. Sullivant (Editors). Emerging applications of algebraic geometry, Vol. 149 of IMA Volumes in Mathematics and its Applications, Springer-Verlag, New York, 2009.

[20] Y. Nesterov. Squared functional systems and optimization problems. Pages 405-440 in H. Frenk, K. Roos, T. Terlaky (Editors). High performance optimization. Kluwer Academic Publishers, Dordrecht, 2000.

[21] Y. Nesterov, A. Nemirovski. Interior-point polynomial algorithms in convex programming. SIAM, Philaldelphia, 1994.

[22] P. A. Parrilo. Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD Thesis, Calif. Inst. Tech., Pasadena, 2000.

[23] M. Putinar. Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J. 42:969-984, 1993.

[24] C. Villani. Topics in optimal transportation. Amer. Math. Society, Providence, 2003.