Control Problems with measures data

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1. Motivation and Model problem
   - Motivation
   - Model problem : minimum time for non-auto. systems

2. A final state constrained control problem with measure data
   - The definition of solution for the state equation
   - Bilateral L^1-viscosity solution

Motivation : a model problem for multi-stage launchers

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Motivation :

We want to steer a *multi-stage* heavy space launcher to the GTO orbit with *minimal fuel consumption* and with a *fixed final mass* (the satellite we leave on the GTO).

(Model problem on a contract CNES-INRIA)

Main ideas of the model : If the launcher is always at full thrust, *minimizing fuel consumption* $\Longleftrightarrow$ *reach GTO in minimal time.*
The mass of the fuel is decreasing w.r. to time $\Longrightarrow$ we can express *the variation of the mass as a function of* $T - t$ (T fixed final time).

O. Bokanowski, A. Briani, H. Zidani *Minimum time control problems for non autonomous differential equations*, Systems & Control Letters, 58, 742-746, 2009.

A minimum time problem

Dynamics

Fix $x \in \mathbb{R}^d$, $T \geq 0$ a control $\alpha \in \mathcal{A} := L^\infty((0,+\infty);A)$, A is a compact convex set of $\mathbb{R}^m$. The trajectory $y_x^{\alpha,T}$ is the solution of :

$$\begin{cases} \dot{y}(t) = f(T-t, y(t), \alpha(t)), & \text{for } t \in (0,T) \\ y(0) = x. \end{cases}$$

Minimum time function

The target $\mathcal{C}$ is a closed convex subset of $\mathbb{R}^d$. We look for

$$\mathcal{T}(x) := \inf_{T \in \mathcal{R}(x)} T,$$

where $\mathcal{R}(x) := \{T \geq 0, \text{ such that } \exists \alpha \ : y_x^{\alpha,T}(T) \in \mathcal{C}\}$.

HJB approach : Dynamic Programming Principle (DPP) ?

For an initial state $x \in \mathbb{R}^d$, an intermediate time $0 \leq h \leq \mathcal{T}(x)$, do we have

$$\mathcal{T}(x) \overset{?}{=} \inf_{T \in \mathcal{R}(x)} \mathcal{T}(y_x^{\alpha,T}(h)) + h$$

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The autonomous case

When the dynamic f is independent of time the DPP HOLDS !
The HJB approach works well both theoretically and numerically.
[M. Bardi, M.G. Crandall, M. Falcone, P.-L. Lions S. Osher, C.-W. Shu, ..]

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In our case : $$\mathcal{T}(x) \geq \inf_{T \in \mathcal{R}(x)} \mathcal{T}(y_x^{\alpha,T}(h)) + h$$

but the inequality can be strict ! (Examples)

What to do ?

We define the reachability value function :
Let φ be the function that takes the value 0 on $\mathcal{C}$ and 1 otherwise.
For each (x, T) we set

$$v(x, T) := \min_{\alpha \in \mathcal{A}} \varphi(y_x^{\alpha, T}(T)).$$

THEN :

1. We can reconstruct $\mathcal{T}(x)$:

$$\mathcal{T}(x) = \min \{ T \geq 0 \text{ such that } v(x, T) = 0 \} .$$

2. v fulfills a Dynamic Programming Principle.

The optimal control problem

For $g_i(t, Y, \alpha)$, $i = 1, \ldots, M$ t-measurable, uniformly Y-Lipschitz and bounded. $X = \mathbb{R}^N$.

$$\begin{cases} \dot{Y}(t) = g_0(t, Y(t), \alpha(t)) & \text{for } t \in (\tau, T] \\ Y(\tau^-) = X \end{cases}$$

The value function

$$v(X, \tau) := \inf_{\alpha \in \mathcal{A}} \varphi(Y^{\alpha}_{X,\tau}(T))$$

φ is lower semi continuous $\mathcal{A} = L^\infty(\mathbb{R}; A)$, A compact.

The optimal control problem

For $g_i(t, Y, \alpha)$, $i = 1, \dots, M$ t-measurable, uniformly Y-Lipschitz and bounded. $X = \mathbb{R}^N$. A Radom measure $\mu = (\mu_1, \dots, \mu_M)$.

$$\begin{cases} \dot{Y}(t) = g_0(t, Y(t), \alpha(t)) + \sum_{i=1}^{M} g_i(t, Y(t), \alpha(t)) d\mu_i & \text{for } t \in (\tau, T] \\ Y(\tau^-) = X \end{cases}$$

We can define a unique solution $Y^{\alpha}_{X,\tau}(t) \in BV([0, T]; \mathbb{R}^N)$.

The value function

$$v(X, \tau) := \inf_{\alpha \in \mathcal{A}} \varphi(Y^{\alpha}_{X,\tau}(T))$$

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HJB approach ?

Dynamic Programming Principle holds :

$$v(X,\tau) = \inf_{a\in A} v(Y^{\alpha}_{X,\tau}(h), h) \qquad \tau \leq h \leq T.$$

Formally HJB :

$$-v_t(X,t) +$$

$$+\sup_{a\in A}\left\{ -D_x v(X,t)\cdot g_0(t,X,a) - D_x v(X,t)\cdot \sum_{i=1}^{M} g_i(t,X,a)\,d\mu_i \right\} = 0$$

with final condition $v(X,T) = \varphi(X) \qquad X \in \mathbb{R}^N.$

What is the meaning of $D_x v(X,t)\cdot\mu$ (V is not smooth) ?

Graph completion (G. Dal Maso- F.Rampazzo, 1991)

- $B \in BV^-((0,T);\mathbb{R}^M)$ is a primitive of μ and $\mathcal{T} := \{t_i,\ i \in \mathbb{N}\}$ the subset of all the discontinuity points of B.
- $(\psi_t)_{t\in\mathcal{T}} := (\psi_t^1,\dots,\psi_t^M)$ be a family of Lipschitz continuous maps :

$$\sum_{t\in\mathcal{T}} V_0^1(\psi_t) < \infty \text{ and } \psi_t(0) = B(t^-) \quad \psi_t(1) = B(t^+) \quad \forall t \in \mathcal{T}$$

We define $\mathcal{W} : [0,T] \to [0,1]$:

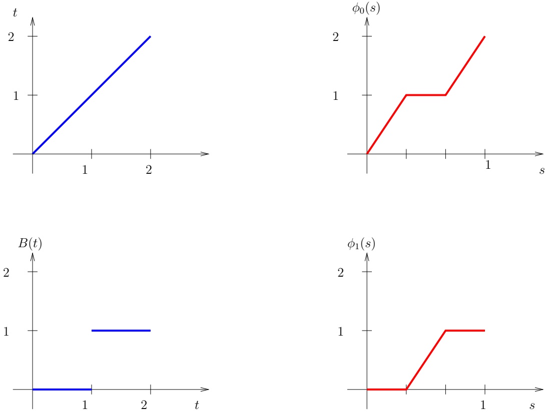
$$\mathcal{W}(t) := \frac{1}{(1+\sum_{i=1}^{+\infty} a_i)}\left(\frac{t + V_0^t(B)}{T + V_0^T(B)} + \sum_{t_i<t} V_0^1(\psi_{t_i})\right).$$

The graph complention of B corresponding to $(\psi_t)_{t\in\mathcal{T}}$ is :

$$(\phi_0,\phi_1,\dots,\phi_M)(s) = \begin{cases} (t, B(t)) & \text{if } s = \mathcal{W}(t);\ t \in [0,T]\setminus\mathcal{T} \\ (t_i, \psi_{t_i}\left(\frac{s-\mathcal{W}(t_i)}{\mathcal{W}(t_i^+)-\mathcal{W}(t_i)}\right)) & \text{if } s \in [\mathcal{W}(t_i),\mathcal{W}(t_i^+)]\ t_i \in \mathcal{T} \end{cases}$$

Graph completion : an example

Ex. : $\mu = \delta_1, t \in [0,2]$



For $\alpha \in \mathcal{A}$, let $Z^{\alpha}_{X,\sigma} : [0,1] \to \mathbb{R}^N$ be the solution of ($\mu = \mu^a dt + \mu^s$)

$$\begin{cases} \dfrac{dZ}{ds}(s) = & \displaystyle\sum_{i=1}^{M} g_i(\phi_0(s), Z(s), \alpha(\phi_0(s))) \left(\mu^a(\phi_0(s)) \frac{d\phi_0}{ds}(s) + \frac{d\phi_i}{ds}(s) \right) + \\ & + g_0(\phi_0(s), Z(s), \alpha(\phi_0(s))) \dfrac{d\phi_0}{ds}(s) \quad \text{for } s \in (\sigma, 1] \\ Z(\sigma) = & X \end{cases}$$

Theorem (G. Dal Maso, F. Rampazzo 91 ; A.B. 1999)

Let μ be a Radom measure and $(\psi_t)_{t\in\mathcal{T}}$ given as before. Then $Y^{\alpha}_{X,\tau} \in BV([\tau, T]; \mathbb{R}^N)$ is a solution in the "sense of measures" if and only if there exists a solution $Z^{\alpha}_{X,\sigma} \in AC([\sigma, 1]; \mathbb{R}^N)$ such that

$$Z(\mathcal{W}(t)) = Y(t) \quad \forall t \in [\tau, T]$$

- Under commutativity assumptions on the g_i the definition does not depent on $(\psi_t)_{t\in\mathcal{T}}$. (A. Bressan & F. Rampazzo 91, 94)
- There are different definitions. (J.P. Raymond 97).

The "reparametrised" optimal control problem

Define the value function for Z :

$$\bar{v}(X,\sigma) = \inf_{\alpha \in \mathcal{A}} \phi(Z^{\alpha}_{X,\sigma}(1)).$$

By the previous theorem we have :

$$v(X,\tau) = \bar{v}(X, W(\tau)) \qquad W(t) \text{ "inverse" of } \phi_0(s) \quad \Rightarrow \textit{Known!}$$

The HJB equation for $\bar{v}$ is without measure :

$$\begin{cases} -\bar{v}_\tau(z,\tau) + \sup_{a\in A}\Big\{ - D_z\bar{v}(z,\tau)\cdot \mathcal{F}(\tau,z,a)\Big\} & = & 0 \\ \bar{v}(z,T) & = & \varphi(z) \end{cases}$$

$\mathcal{F}(\tau,z,a)$ is τ-measurable and φ is lower semi continuous,
$\Rightarrow$ NOT classical viscosity solution !

In : A. Briani & H. Zidani *Characterisation of the value function of final state constrained control problems with measures data*, (in preparation).

New definition : Bilateral L^1 viscosity solution

A bounded *l.s.c. function u is a Bilateral L^1-viscosity solution* if : for any $b \in L^1(0,T)$, $\phi \in C^1(\mathbb{R}^N)$ and (x_0, t_0) local minimum point for $u(x,t) - \int_0^t b(s)ds - \phi(x)$ we have

$$\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-t_0|\leq\delta} \sup_{x\in B_\delta(x_0),\, p\in B_\delta(D\phi(x_0))} \{H(t,x,p) + b(t)\} \geq 0$$

and

$$\lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|t-t_0|\leq\delta} \inf_{x\in B_\delta(x_0),\, p\in B_\delta(D\phi(x_0))} \{H(t,x,p) + b(t)\} \leq 0.$$

Bilateral : E.N. Barron & R. Jensen 1990-1991 ; G. Barles & B. Perthame 1987 ; L^1-viscosity solution : H. Ishii 1985 ; P.L. Lions & B. Perthame 1987 ; D. Nunziante 1990, 1992 ; A.B. & F. Rampazzo 2005 ; G Barles 2006 ; M. Bourgoing 2008.

- Consistency, stability and uniqueness.
- The value function $\bar{v}$ is a Bilateral L^1-viscosity solution.

CONVERGENCE RESULT :

Let $\varphi_m \in C^0$ and $\varphi_m(x) \uparrow \varphi(x)$, $\Delta_k := (\Delta x_k, \Delta t_k)$ and denote by $v_m^{\Delta_k}$ the solution of

$$S(t_{n+1}, x_j, v_j^{n+1}, v^n) = 0 \qquad v(x_j, 0) = \varphi_m(x_j).$$

If the scheme S is *Monotone, Stable and "Consistent"*.
Then, as $k \to 0$, $m \to \infty$, $v_m^{\Delta_k}$ converges pointwise to the value function $\bar{v}$. (An example of finite difference type S is given.)

$\Rightarrow$ We can calculate $\bar{v}$ and by $v(X, \tau) = \bar{v}(X, W(\tau))$ obtain v !

....thanks for your attention.