Convergence d'une méthode de résolution de systèmes d'inclusions monotones couplées

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Notation

Let $\mathcal{H}$ be a Hilbert space. $\Gamma_0(\mathcal{H})$ denotes the class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty, +\infty]$.
For an operator $A \colon \mathcal{H} \to 2^{\mathcal{H}}$

- $\operatorname{gra}(A) = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\big\}$ is the graph of A.
- A is called monotone if, for every (x,u) and (y,v) in $\operatorname{gra}(A)$, we have

  $$\langle x - y \mid u - v\rangle \geq 0.$$

  In addition, is maximal if its graph is maximal (in the sense of inclusions) between the monotone operators in $\mathcal{H}$.
- The resolvent operator of A, $J_A = (\operatorname{Id} + A)^{-1}$, is single-valued.

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Motivation

Let C_1 y C_2 be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$.

Best approximation problem

$$\min_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\|^2$$

$$\begin{cases} 0 \in N_{C_1} x_1 + x_1 - x_2 \\ 0 \in N_{C_2} x_2 - x_1 + x_2 \end{cases}$$

is equivalent to

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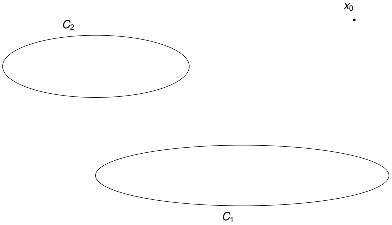

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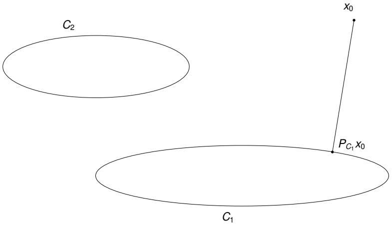

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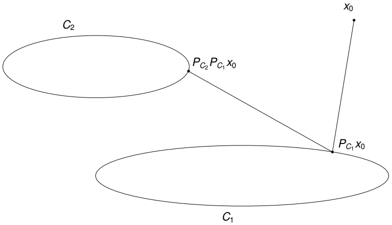

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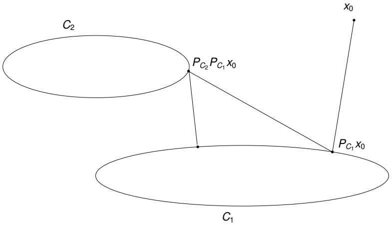

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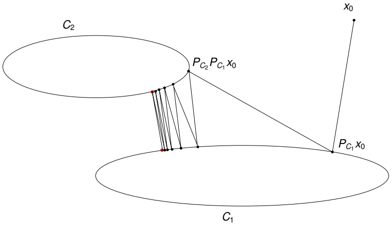

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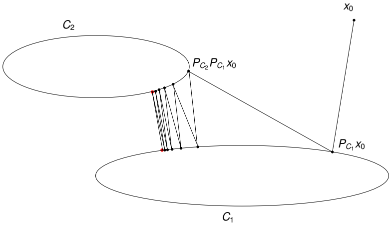

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Let f_1 y f_2 in $\Gamma_0(\mathcal{H})$.

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$$(\mathcal{P}_1) \qquad \min_{x_1 \in \mathcal{H},\, x_2 \in \mathcal{H}} f_1(x_1) + f_2(x_2) + \frac{1}{2}\|x_1 - x_2\|^2$$

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Let $x_{2,0} \in \mathcal{H}$ and set

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- Let A_1, A_2 be maximal monotone operators defined on a Hilbert space $\mathcal{H}$.

Monotone inclusion

$$(\mathcal{P}_2) \quad \begin{cases} 0 \in A_1 x_1 + x_1 - x_2 \\ 0 \in A_2 x_2 - x_1 + x_2 \end{cases}$$

- Let $x_{1,0} \in \mathcal{H}$. For all $n \in \mathbb{N}$ $\begin{cases} x_{2,n} = (\mathrm{Id} + A_2)^{-1} x_{1,n} \\ x_{1,n+1} = (\mathrm{Id} + A_1)^{-1} x_{2,n}. \end{cases}$
- $x_{1,n} \rightharpoonup x_1$ and $x_{2,n} \rightharpoonup x_2$ where (x_1, x_2) is solution of $(\mathcal{P}_2)$ (Bauschke et al.'05).

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Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{G}$ be real Hilbert spaces, and, for every $i \in \{1,2\}$, let $f_i \in \Gamma_0(\mathcal{H}_i)$ and let $L_i \colon \mathcal{H}_i \to \mathcal{G}$ be linear and bounded.

Attouch et al.'08

$$(\mathcal{P}_3) \qquad \min_{x_1 \in \mathcal{H}_1,\, x_2 \in \mathcal{H}_2} f_1(x_1) + f_2(x_2) + \frac{1}{2}\|L_1 x_1 - L_2 x_2\|^2$$

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General Problem

- For every $i \in \{1, \dots, m\}$:
  - $\mathcal{H}_i$ is a real Hilbert space and denote $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$.
  - A_i is a maximal monotone operator from $\mathcal{H}_i$ to $2^{\mathcal{H}_i}$
  - B_i is an operator from $\boldsymbol{\mathcal{H}}$ to $\mathcal{H}_i$ (coupling) such that $(B_i)_{1 \le i \le m}$ satisfy

$$(\forall (x_1, \dots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m)(\forall (y_1, \dots, y_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m)$$

$$\sum_{i=1}^{m} \langle B_i(x_1, \dots, x_m) - B_i(y_1, \dots, y_m) \mid x_i - y_i \rangle$$

$$\geq \beta \sum_{i=1}^{m} \left\| B_i(x_1, \dots, x_m) - B_i(y_1, \dots, y_m) \right\|^2 \quad (C)$$

for some $\beta \in \left]0, +\infty\right[$

General Problem

Un-coupled equilibrium problem

Find $(x_i)_{1\leq i\leq m} \in \mathcal{H}$ such that

$$\begin{cases} 0 & \in A_1 x_1 \\ 0 & \in A_2 x_2 \\ & \vdots \\ 0 & \in A_m x_m \end{cases}$$

General Problem

Coupled equilibrium problem

Find $(x_i)_{1\leq i\leq m} \in \mathcal{H}$ such that

$$(P)\begin{cases} 0 & \in A_1 x_1 + B_1(x_1,\ldots,x_m) \\ 0 & \in A_2 x_2 + B_2(x_1,\ldots,x_m) \\ & \vdots \\ 0 & \in A_m x_m + B_m(x_1,\ldots,x_m) \end{cases}$$

Algorithm and weakly convergence

- For every $n \in \mathbb{N}$ and every $i \in \{1, \ldots, m\}$, set

Algorithm

$$x_{i,n+1} = \lambda_{i,n} x_{i,n} + (1 - \lambda_{i,n})\Big(J_{\gamma_n A_i}\big(x_{i,n} - \gamma_n B_i(x_{1,n}, \ldots, x_{m,n})\big)\Big)$$

- where for every $i \in \{1, \ldots, m\}$
  - $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0, 2\beta[$
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  - $x_{i,0} \in \mathcal{H}_i$
  - $(\lambda_{i,n})_{n\in\mathbb{N}}$ be a sequence in $[0, 1[$ such that $\sum_{n\in\mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty$.
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Strong convergence

Suppose that, for some $i \in \{1, \ldots, m\}$, one of the following holds

- $\mathcal{H}_i$ is finite dimensional
- A_i is uniformly monotone, i.e. there exists an nondecreasing function $\phi\colon [0, +\infty[\to [0, +\infty]$, which vanishes only at 0, such that

$$(\forall (x, u) \in \operatorname{gra} A_i)(\forall (y, v) \in \operatorname{gra} A_i) \quad \langle x - y \mid u - v \rangle \geq \phi(\|x - y\|)$$

- J_{A_i} is compact
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Then $x_{i,n} \to x_i$.

Strong convergence

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$$\sum_{i=1}^{m} \langle B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m) \mid y_i - x_i \rangle \geq \phi\Big(\sum_{i=1}^{m} \|x_i - y_i\|^2\Big)$$

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Coupling evolution equations

Let $T \in \,]0, +\infty[$ and, for every $i \in \{1, \ldots, m\}$, let H_i be a Hilbert space.
The problem is to find functions $(x_i)_{1 \leq i \leq m}$ in $W^{1,2}([0,T], \mathsf{H}_1) \times \cdots \times W^{1,2}([0,T], \mathsf{H}_m)$ such that, a.e. in $t \in \,]0,T[$

Coupling evolution problem (CEP)

$$(\forall i \in \{1, \ldots, m\}) \quad \begin{cases} -x_i'(t) \in \partial \mathsf{f}_i(x_i(t)) + D_i(x_1(t), \ldots, x_m(t)); \\ x_i(0) = x_i(T), \end{cases}$$

where, for every $i \in \{1, \ldots, m\}$,

- $\mathsf{f}_i \in \Gamma_0(\mathsf{H}_i)$
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For every $i \in \{1, \dots, m\}$, let

- $\mathcal{H}_i = L^2([0, T]; \mathsf{H}_i)$,
- $\mathcal{W}_i = \{x \in \mathcal{C}([0, T]; \mathsf{H}_i) \cap W^{1,2}([0, T]; \mathsf{H}_i) \mid x(T) = x(0)\}$,
- $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$,

$$A_i x = \begin{cases} \{u \in \mathcal{H}_i \mid u(t) \in x'(t) + \partial \mathsf{f}_i(x(t)) \text{ a.e. in }]0, T[\,\}, & \text{if } x \in \mathcal{W}_i; \\ \varnothing, & \text{otherwise.} \end{cases}$$

is maximal monotone,

- $B_i \colon \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i$,

$$B_i(x_1, \dots, x_m) \colon [0, T] \to \mathsf{H}_i$$
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Coupling evolution equations

Problem (CEP) is equivalent to (P) and Algorithm becomes (we set errors to 0 and $\lambda_n \equiv 0$), for every $i \in \{1,\ldots,m\}$ and a.e. in $t \in]0,T[$,

Algorithm (CEP)

$$(A) \quad x_{i,n}(t)/\gamma_n - D_i(x_{1,n}(t),\ldots,x_{m,n}(t)) \in x'_{i,n+1}(t) + \partial \mathrm{f}_i(x_{i,n+1}(t)) + x_{i,n+1}(t)/\gamma_n,$$

where

- $(\gamma_n)_{n\in\mathbb{N}}$ in $]0,2\beta[$
- $(\forall i \in \{1,\ldots,m\}) \quad x_{i,0} \in W^{1,2}([0,T];\mathrm{H}_i)$.

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There exists an unique solution of the inclusion (A) (Brezis'73).

The algorithm converges weakly to the solution of (CEP).

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Network Flows

- Directed graph (transport network).
- m types of flows, M links, N paths, Q origin-destination pairs. L is a $M \times N$ binary matrix (L_{jl} is 1 if link j belongs to path l and 0 otherwise).
- For every $i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, Q\}$
  - $x_i \in \mathbb{R}^N$: flow type i on all the routes.
  - d_{ik}: total flow type i that transit from the origin to the destination of k.
  - N_k: number of paths linking k.
  - $C_i = \{(\eta_l)_{1 \leq l \leq N} \in [0, +\infty[^N \mid \forall k \ \sum_{l \in N_k} \eta_l = d_{ik}\}$
- For every $j \in \{1, \ldots, M\}$,
  - $\phi_j \colon \mathbb{R} \to [0, +\infty[$: strictly increasing τ-Lipschitz continuous function (cost of transiting on link j).
  - $h_j \colon \mathbb{R}^{Nm} \to \mathbb{R} \colon (x_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m (Lx_i)^\top e_j$ total flow through link j.

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Network Flows

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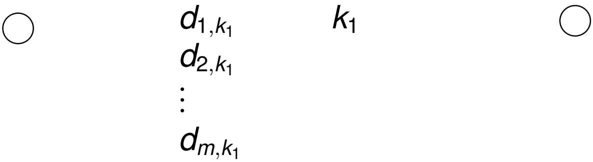

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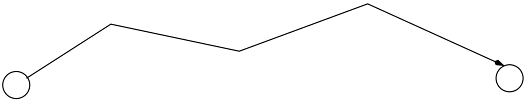

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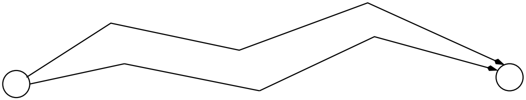

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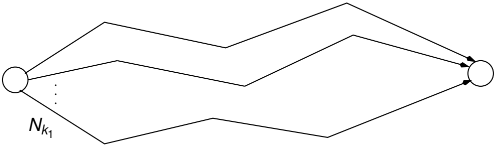

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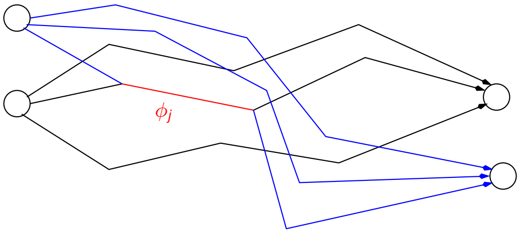

Network Flows

The Wardrop equilibria are the solutions of

Beckmann et al.'56

$$(N) \quad \underset{x_1 \in C_1, \ldots, x_m \in C_m}{\text{minimize}} \sum_{j=1}^{M} \int_0^{h_j(x_1,\ldots,x_m)} \phi_j(h)dh.$$

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(N) is a particular case of (P). Then our Algorithm, which becomes

$\forall i \quad x_{i,n+1} = P_{C_i}(x_{i,n} - \gamma L^\top(\phi_1(\rho_{1,n}), \ldots, \phi_m(\rho_{m,n}))),$

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Thank you for your attention !