Total Variation Projection: Algorithms and Applications

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Overview

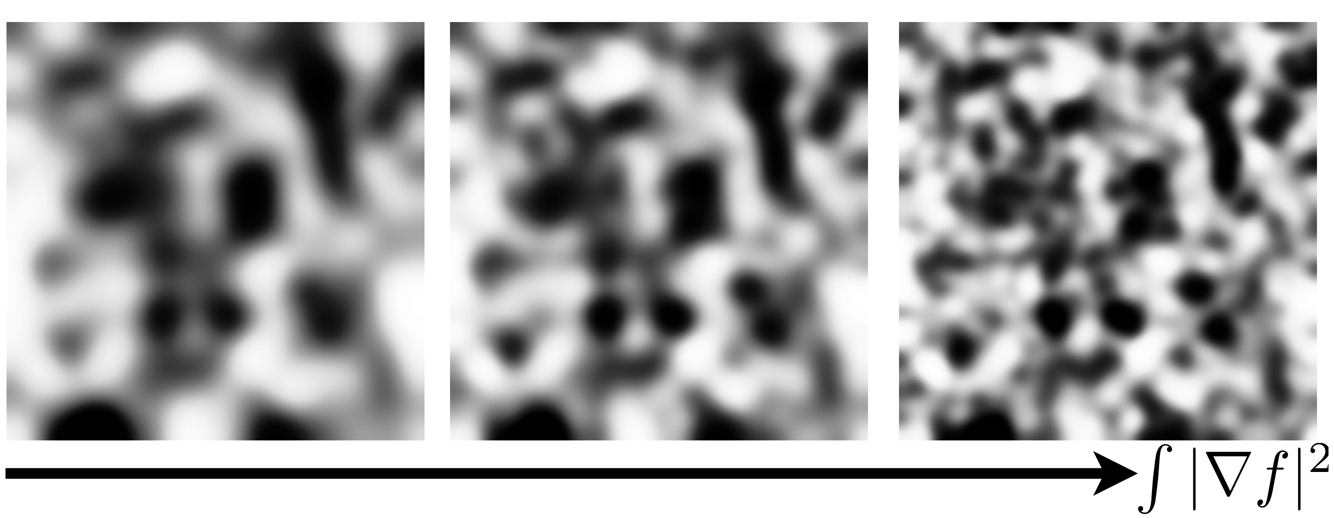
- Total Variation Prior
- Primal and Dual Projection
- First Order Algorithms
- Inverse Problem Resolution by Projection
- Cheeger Set Approximation

Smooth and Cartoon Priors

Prior model: energy $J(f) \in \mathbb{R}$ low for images of the model $f \in \Theta$.

Sobolev norm:

$$J(f) = \frac{1}{2}\|f\|^2_{\text{Sob}} = \frac{1}{2}\int \|\nabla_x f\|^2 \mathrm{d}x$$



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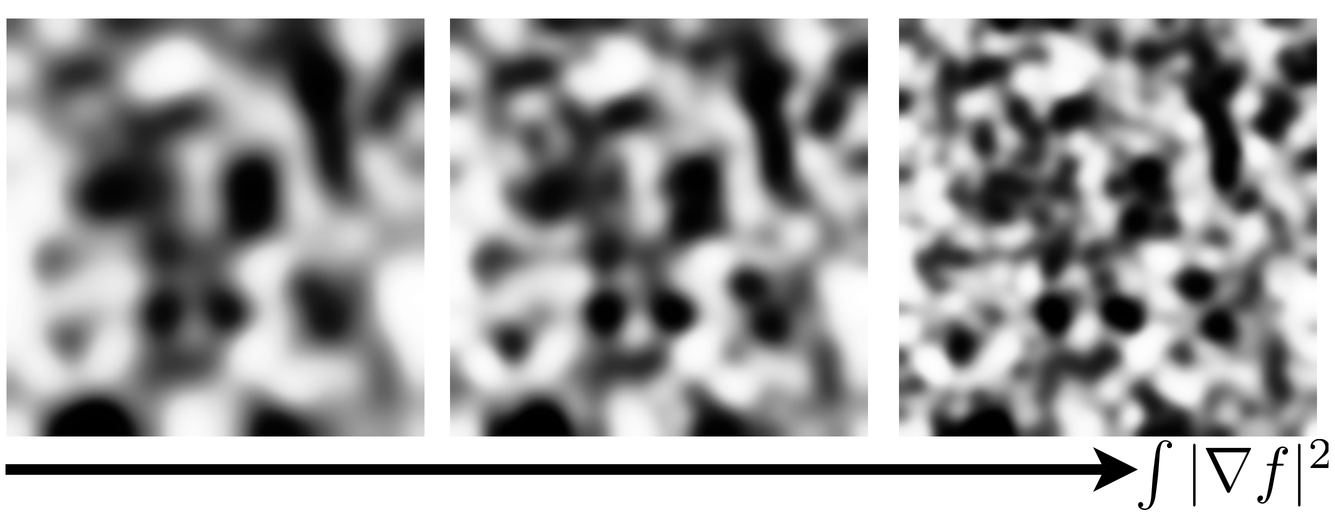
Sobolev norm: $$J(f) = \frac{1}{2}\|f\|^2_{\text{Sob}} = \frac{1}{2}\int \|\nabla_x f\|^2 \mathrm{d}x$$

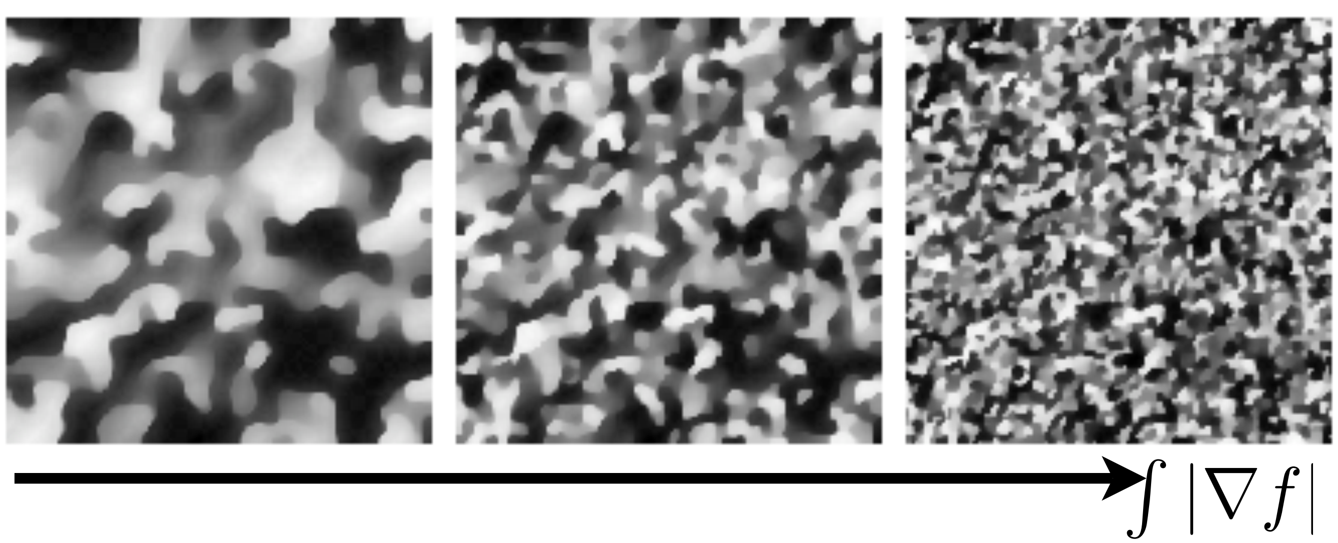
Total variation norm: $$J(f) = \|f\|_{\text{TV}} = \int \|\nabla_x f\| \mathrm{d}x$$

$\longrightarrow$ Extension to non-smooth functions $f \in \text{BV}([0,1]^2)$

Co-area formula: $$\|f\|_{\text{TV}} = \int_{\mathbb{R}} \text{length}(\mathcal{C}_t)\mathrm{d}t$$

Level set $\mathcal{C}_t = \{x \setminus f(x) = t\}$

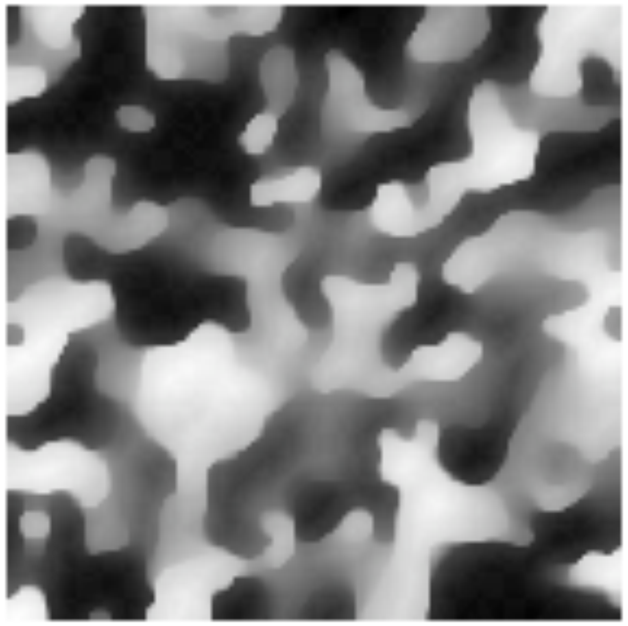




Natural Image Priors

"Typical" image drawn at random: (denoising noise)

Small $\|f\|_{\text{Sob}} = \int \|\nabla f\|^2$



Small $\|f\|_{\text{TV}} = \int \|\nabla f|$

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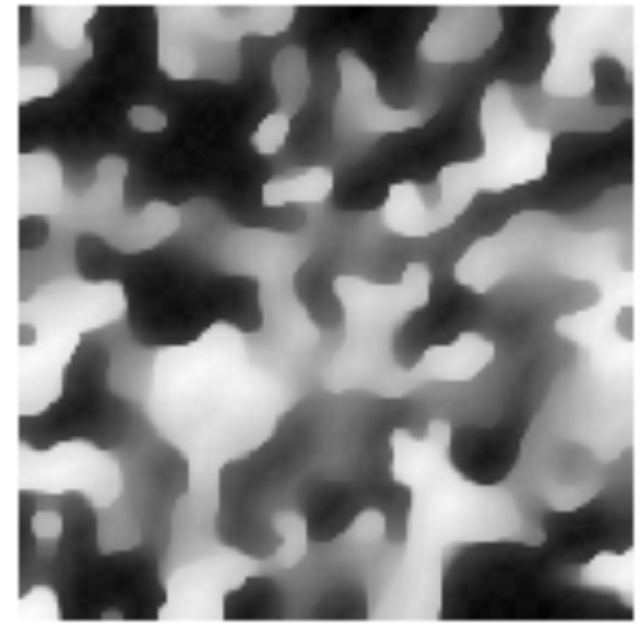
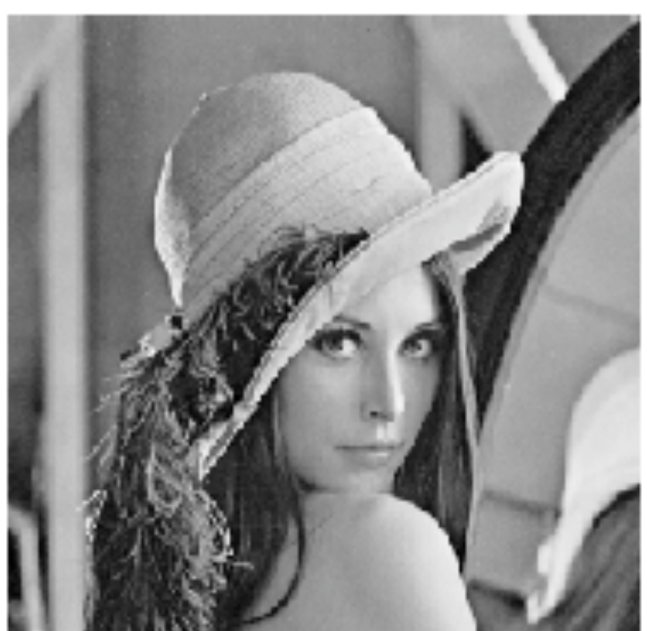


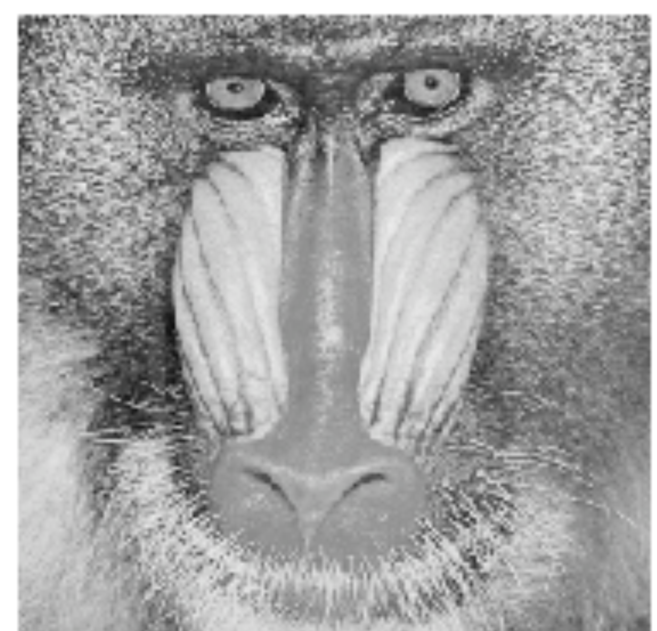
Small $\|f\|_{\text{TV}} = \int \|\nabla f|$

Natural images: structure + texture + noise + ...

TV=3988



TV=9387



Discrete Priors

Analog signal $f \in \mathrm{L}^2([0,1]^2) \longrightarrow$ discrete signal $f \in \mathbb{R}^N$.

Finite differences operators:

$$\delta_1 f[n_1, n_2] = f[n_1+1, n_2] - f[n_1, n_2]$$

$$\delta_2 f[n_1, n_2] = f[n_1, n_2+1] - f[n_1, n_2]$$

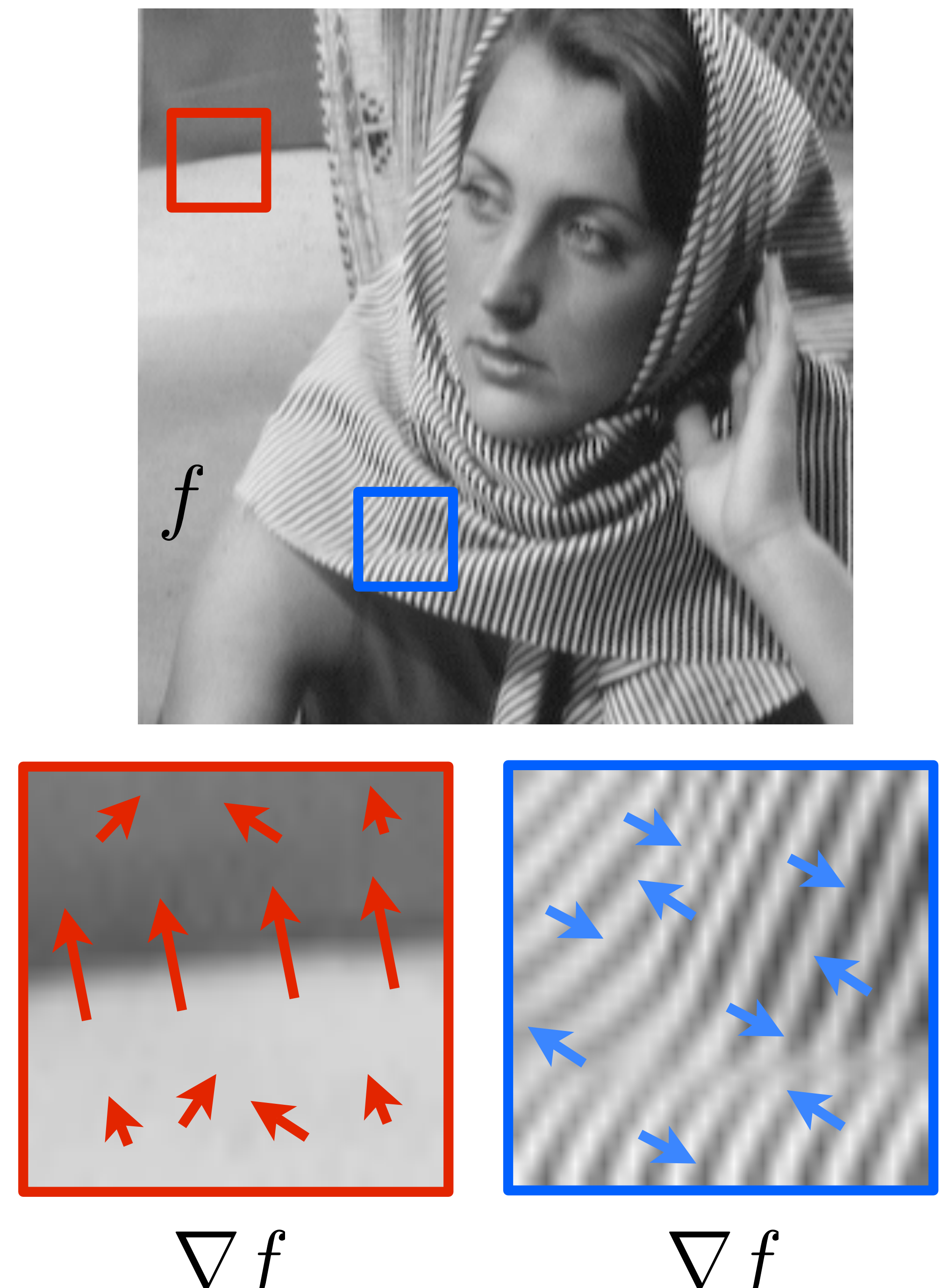
Discrete gradient:

$$\nabla f[n] = (\delta_1 f[n], \delta_2 f[n]) \in \mathbb{R}^{2\times N}$$

Discrete energies:

$$J_{\text{Sob}}(f) = \frac{1}{2} \sum_n (\delta_1 f[n])^2 + (\delta_2 f[n])^2$$

$$J_{\text{TV}}(f) = \sum_n \sqrt{(\delta_1 f[n])^2 + (\delta_2 f[n])^2}$$



Overview

- Total Variation Prior
- **Primal and Dual Projection**
- First Order Algorithms
- Inverse Problem Resolution by Projection
- Cheeger Set Approximation

Denoising by Regularization

Noisy observations: $f = f_0 + w$.

$$f^\star_\lambda = \underset{g \in \mathbb{R}^N}{\text{argmin}} \; \frac{1}{2}\|f - g\|^2 + \lambda \|g\|_{\text{TV}}$$

L^2-*constraint:*
$$f^\star_\varepsilon = \underset{\|f-g\| \leqslant \varepsilon}{\text{argmin}} \; \|g\|_{\text{TV}}$$

TV projection:
$$f^\star_\tau = \underset{\|g\|_{\text{TV}} \leqslant \tau}{\text{argmin}} \; \|f - g\|$$
[Combettes,Pesquet]

Equivalence $\lambda \leftrightarrow \varepsilon \leftrightarrow \tau$.

Best formulation: depends on the prior knowledge!

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Today's focus

Equivalence $\lambda \leftrightarrow \varepsilon \leftrightarrow \tau$.

Best formulation: depends on the prior knowledge!

Dual Formulation

$$f^\star = \underset{\|g\|_{\mathrm{TV}} \leqslant \tau}{\operatorname{argmin}} \|f - g\| = \underset{g}{\operatorname{argmin}} \frac{1}{2}\|f - g\|^2 + 1_{\|\cdot\|_{\mathrm{TV}} \leqslant \tau}(g)$$

Dual Formulation

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Dual vector field $u = \{u_i\}_{i=0}^{N-1} \in \mathbb{R}^{N \times 2}$ where $u_i \in \mathbb{R}^2$.

TV norm: $\|f\|_{\mathrm{TV}} = \|u\|_1$ where $u = \nabla f$. $\|u\|_1 = \sum_{i=0}^{N-1} \sqrt{u_{i,1}^2 + u_{j,2}^2}$

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ℓ^1–ℓ^∞ duality:

$$1_{\|\cdot\|_1 \leqslant \tau}(v) = \max_u \langle u,\, v\rangle - \tau \|u\|_\infty \qquad \|u\|_\infty = \max_{0 \leqslant i < N} \sqrt{u_{i,1}^2 + u_{j,2}^2}$$

$$\Longrightarrow \quad 1_{\|\cdot\|_{\mathrm{TV}} \leqslant \tau}(f) = \min_u \langle \operatorname{div}(u),\, f\rangle + \tau \|u\|_\infty$$

Proposition: one has $\quad f^\star = f - \operatorname{div}(u^\star)$

$$u^\star = \underset{u \in \mathbb{R}^{N\times 2}}{\operatorname{argmin}} \frac{1}{2}\|f_0 - \operatorname{div}(u)\|^2 + \tau \|u\|_\infty$$

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Proximal Iterations

$$u^\star = \underset{u \in \mathbb{R}^{N\times 2}}{\operatorname{argmin}} \underbrace{\frac{1}{2}\|f_0 - \operatorname{div}(u)\|^2}_{\text{smooth}} + \underbrace{\tau \|u\|_\infty}_{\text{``simple''}}$$

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Forward-backward splitting: *Initialization:* $u^{(0)} = 0,\ k \leftarrow 0.$

Gradient descent: $\tilde{u}^{(k)} = u^{(k)} - \mu \operatorname{grad}\left(f_0 - \operatorname{div}(u^{(k)})\right)$

ℓ^∞ *proximal correction:* $u^{(k+1)} = \operatorname{prox}_{\mu\lambda\|\cdot\|_\infty}(\tilde{u}^{(k)}).$

$$\operatorname{prox}_{\kappa\|\cdot\|_\infty}(u) = \underset{v}{\operatorname{argmin}} \frac{1}{2}\|u - v\|^2 + \kappa\|v\|_\infty.$$

Continue? If $\|u^{(k+1)} - u^{(k)}\| > \eta,\ k \leftarrow k+1.$

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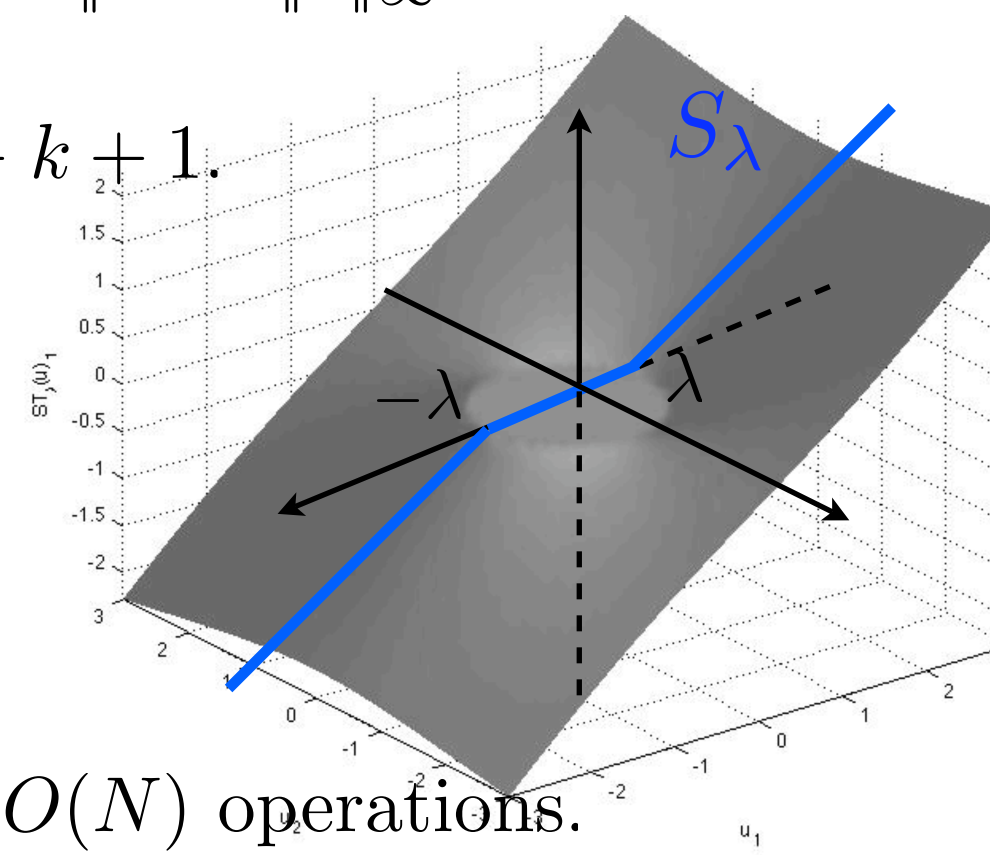
Continue? If $\|u^{(k+1)} - u^{(k)}\| > \eta$, $k \leftarrow k+1$.

Proximal operator computation:

$$\text{prox}_{\kappa\|\cdot\|_\infty}(u) = u - S_\lambda(u)$$

where $$S_\lambda(u)_i = \max\left(1 - \frac{\lambda}{\|u_i\|}, 0\right) u_i$$



Computing λ: partial sorting the values $\|u_i\|$: $O(N)$ operations.

Nesterov Multi-step

Accelerating the gradient descent: use $u^{(\ell)}$ for all $\ell \leqslant k$ to compute $u^{(k+1)}$.

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Nesterov multi-step scheme:

Initialization: $u^{(0)} = 0, A_0 = 0, \xi^{(0)} = 0, k \leftarrow 0.$

First proximal computation: $v^{(k)} = \mathrm{prox}_{A_k \tau \|\cdot\|_\infty}(u^{(0)} - \xi^{(k)})$

Set $a_k = \dfrac{\mu + \sqrt{\mu^2 + 4\mu A_k}}{2}$ $\quad \omega^{(k)} = \dfrac{A_k + a_k v^{(k)}}{A_k + a_k}$

Second proximal computation: $u^{(k+1)} = \mathrm{prox}_{\mu\tau/2\|\cdot\|_\infty}(\tilde{\omega}^{(k)})$

where $\tilde{\omega}^{(k)} = \omega^{(k)} - \frac{\mu}{2}\nabla\left(f_0 - \mathrm{div}(\omega^{(k)})\right)$

Update $A_{k+1} = A_k + a_k$ and

$$\xi^{(k+1)} = \xi^{(k)} + a_k \,\mathrm{grad}\left(f_0 - \mathrm{div}(u^{(k+1)})\right)$$

Continue? If $\|u^{(k+1)} - u^{(k)}\| > \eta$, $k \leftarrow k+1$.

Convergence Analysis

$$u^\star = \underset{u \in \mathbb{R}^{N\times 2}}{\operatorname{argmin}}\; E(u) = \frac{1}{2}\|f_0 - \operatorname{div}(u)\|^2 + \tau\|u\|_\infty$$

Known results: If $0 < \mu < 2/\|\nabla^*\nabla\| = 1/4$,

$$E(u^{(k)}) - E(u^\star) = O(1/k^\alpha) \quad \text{with} \quad \alpha = \begin{cases} 1 & \text{for Forward-Backward} \\ 2 & \text{for Nesterov} \end{cases}$$

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$\longrightarrow$ what about the convergence of $f^{(k)} = f - \text{div}(u^{(k)})$?

Key lemma: [Fadili,Peyré]: $\|f^{(k)} - f^\star\|^2 \leqslant 2\left(E(u^{(k)}) - E(u^\star)\right)$

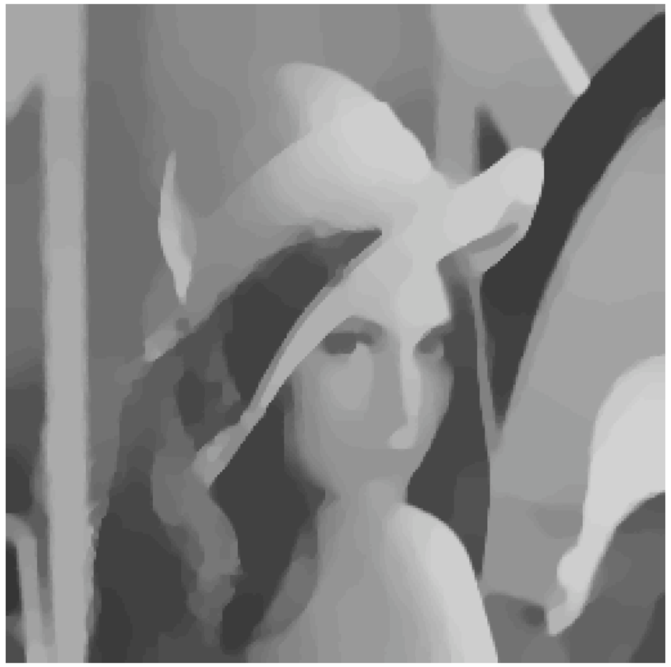
Theorem: [Fadili, Peyré] If $0 < \mu < 1/4$, then

$$\|f^{(k)} - f^\star\|^2 = O(1/k^\alpha) \quad \text{with} \quad \alpha = \begin{cases} 1 & \text{for Forward-Backward} \\ 2 & \text{for Nesterov} \end{cases}$$

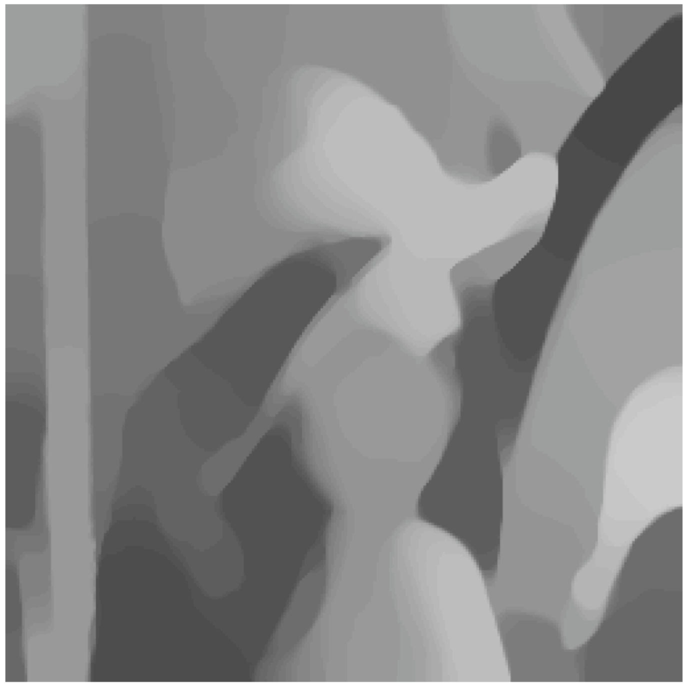
TV-projection Denoising

Noisy image y

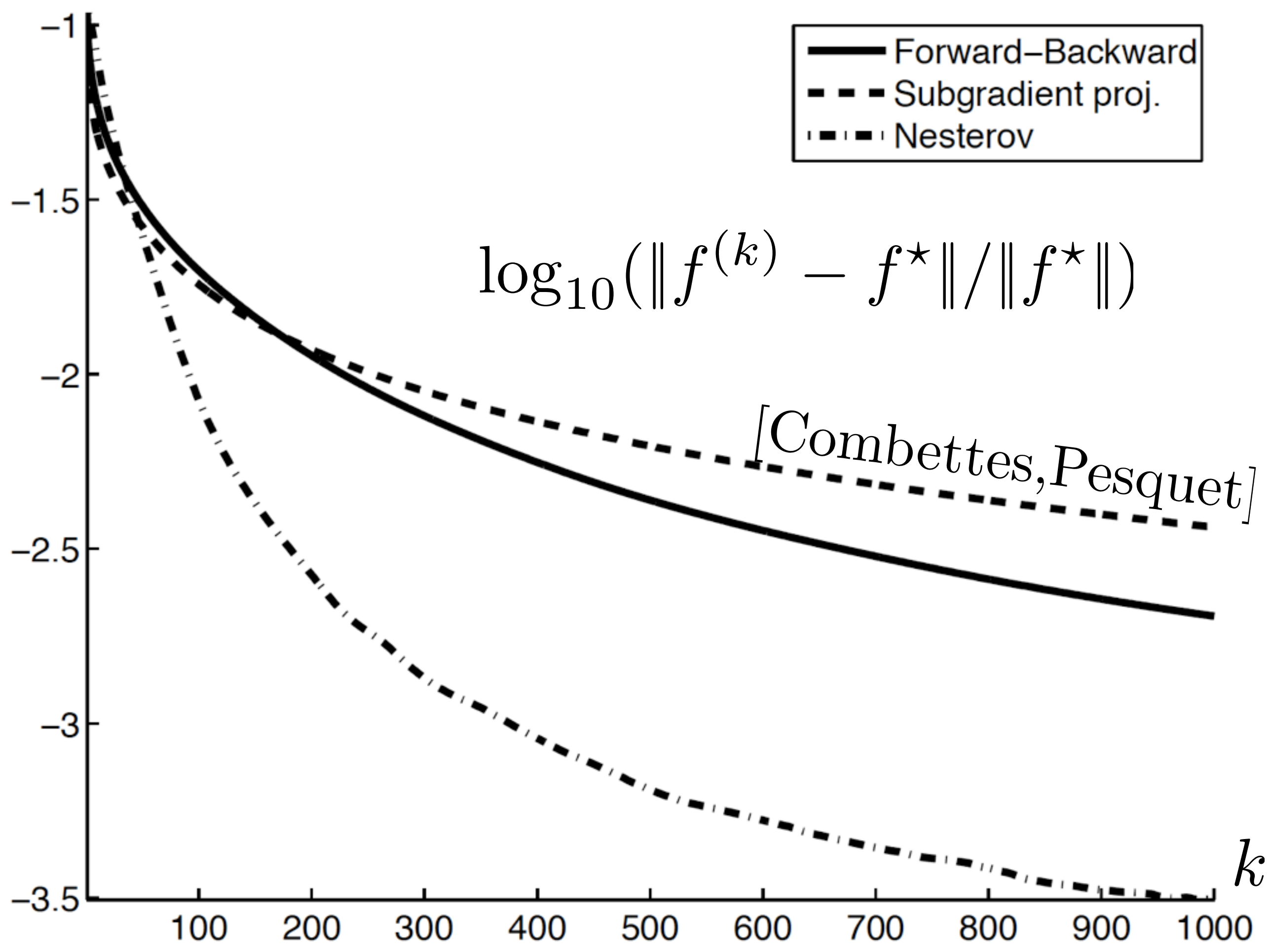
$f^\star$ with $\|f_0\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 2$

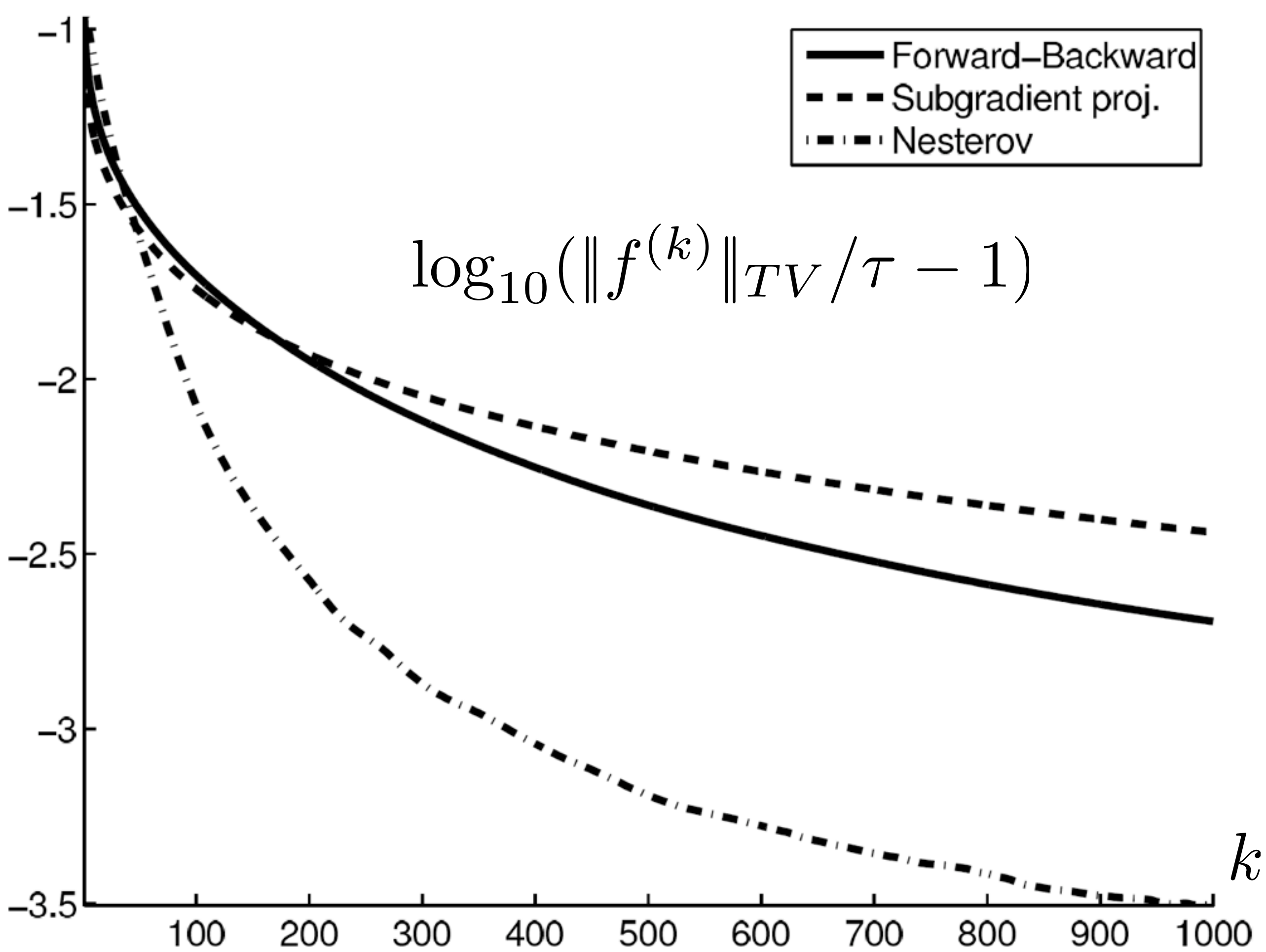


$f^\star$ with $\|f_0\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 4$



$f^\star$ with $\|f_0\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 8$





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Inverse Problems

Recovering f_0 from P noisy measurements $y = \Phi f_0 + w \in \mathbb{R}^P$.

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}^P$ with $P \ll N$ (missing information)

$w[n] \sim \mathcal{N}(0, \sigma)$ white noise.

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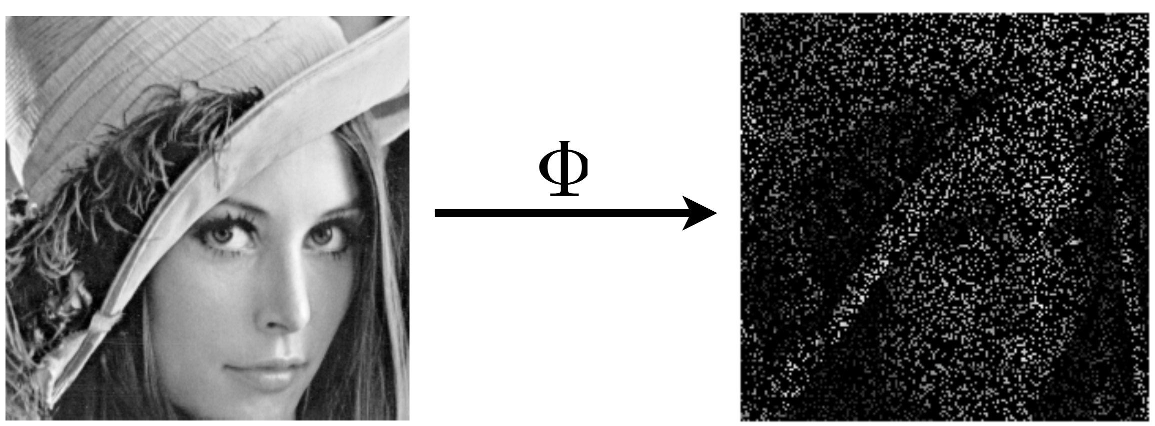
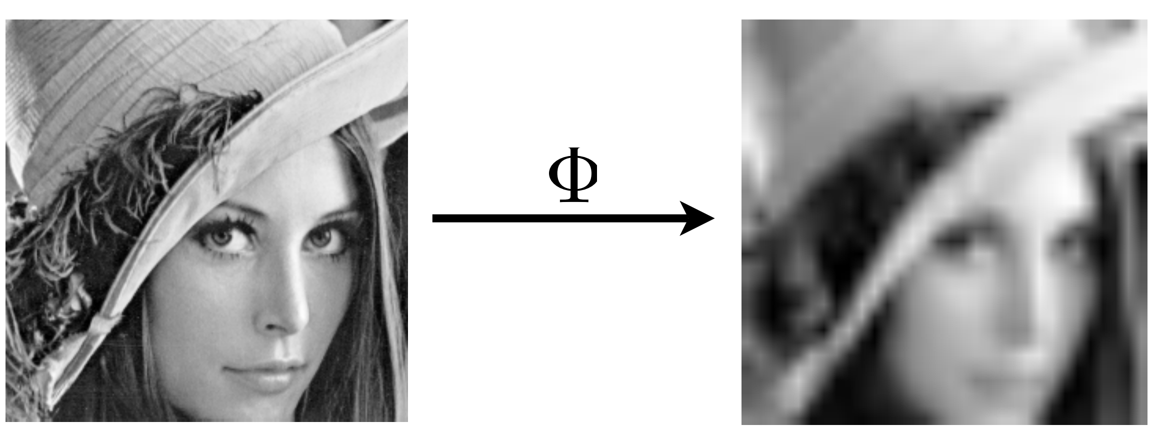
Inpainting: set $\Omega \subset \{0, \ldots, N-1\}$ of missing pixels, $P = N - |\Omega|$.



$$(\Phi f)(x) = \begin{cases} 0 & \text{if} \quad x \in \Omega, \\ f(x) & \text{if} \quad x \notin \Omega. \end{cases}$$

Super-resolution: $\Phi f = (f \ast \varphi) \downarrow_k$, $P = N/k$.



TV Projection for Inverse Problems

Inverse problem resolution: $$f^\star = \underset{\|f\|_{\mathrm{TV}} \leqslant \tau}{\operatorname{argmin}} \frac{1}{2}\|\Phi f - y\|^2$$

Projected gradient descent:

$$f^{(\ell+1)} = \mathrm{Proj}_{\{\|\cdot\|_{\mathrm{TV}} \leqslant \tau\}} \tilde f^{(\ell)} \qquad \text{where} \quad \tilde f^{(\ell)} = f^{(\ell)} + \nu \Phi^*(y - \Phi f^{(\ell)})$$

Theorem: If $\nu \in (0, 2/\|\Phi^*\Phi\|)$, then $\|f^{(\ell)} - f^\star\| = O(1/\ell)$.

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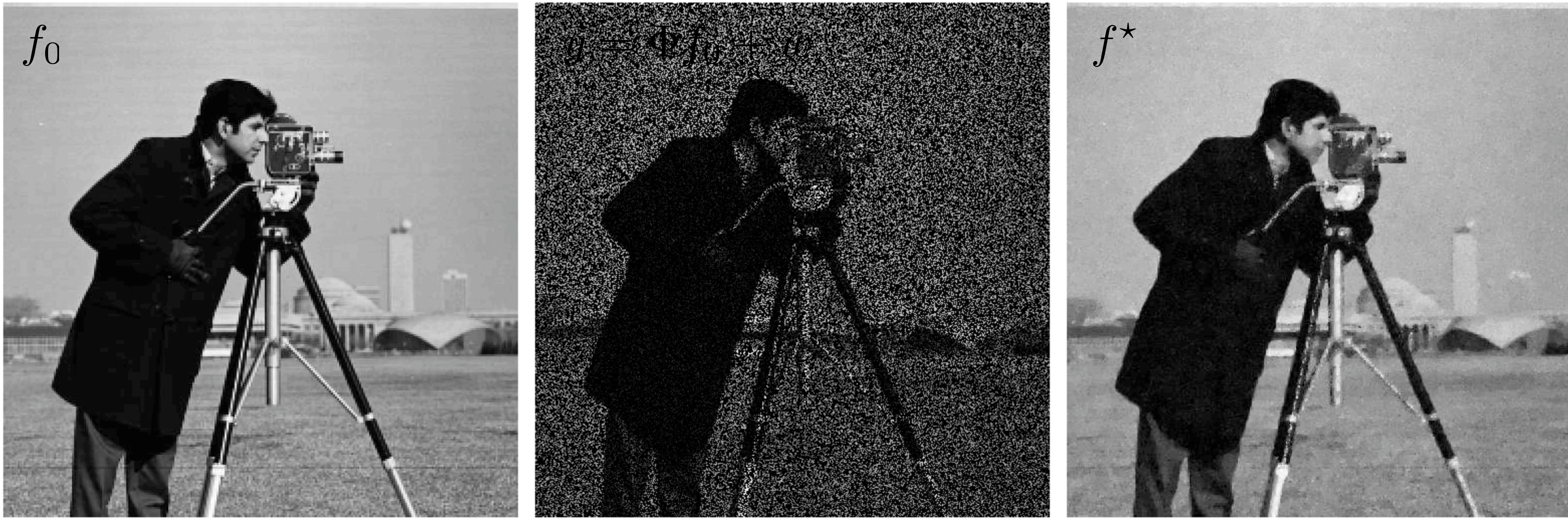
Imperfect projection:

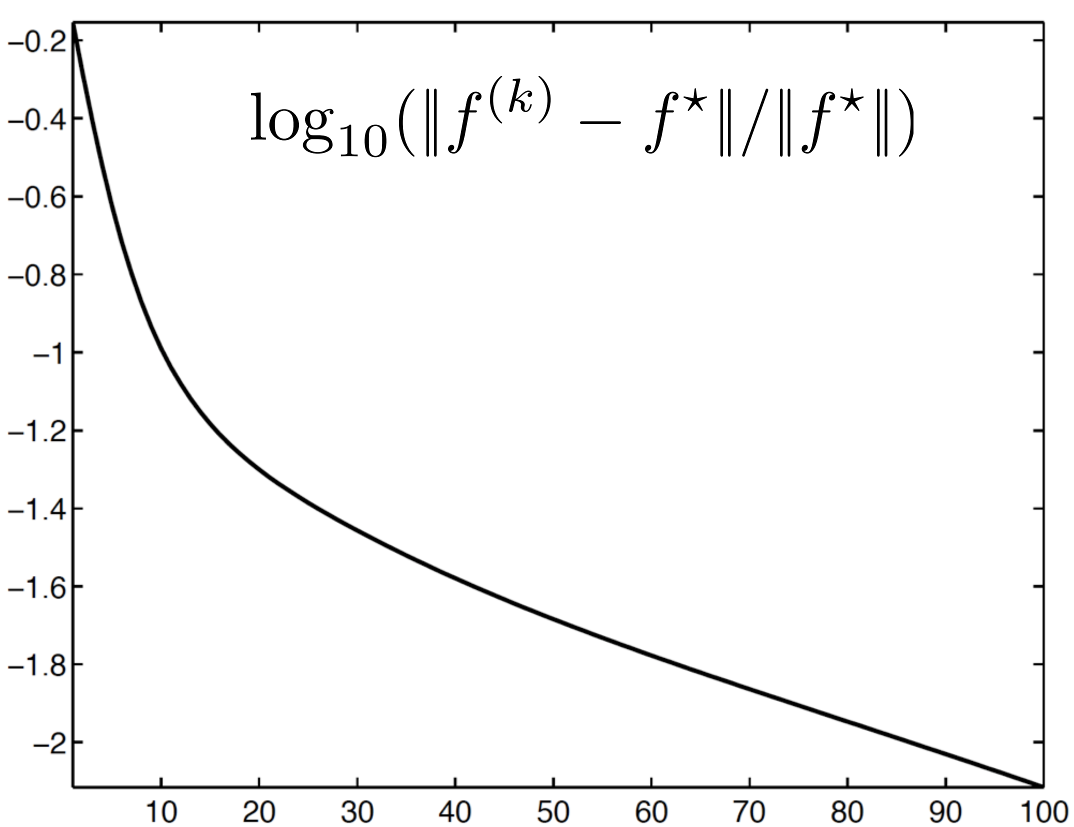
$$f^{(\ell+1)} = \mathrm{Proj}_{\{\|\cdot\|_{\mathrm{TV}} \leqslant \tau\}} \left(f^{(\ell)} + \nu \Phi^*(y - \Phi f^{(\ell)}) \right) + a_\ell$$

Convergence of $f^{(\ell)}$ if $\sum_\ell \|a_\ell\| < +\infty$. (does not work with Nesterov)

Theorem: If $\mathrm{Proj}_{\|\cdot\|_{\mathrm{TV}} \leqslant \tau} \tilde f^{(\ell)}$ is computed with η_ℓ-tolerance and if $\sum_\ell \eta_\ell < +\infty$, then $f^{(\ell)}$ converges to $f^\star$

Inpainting





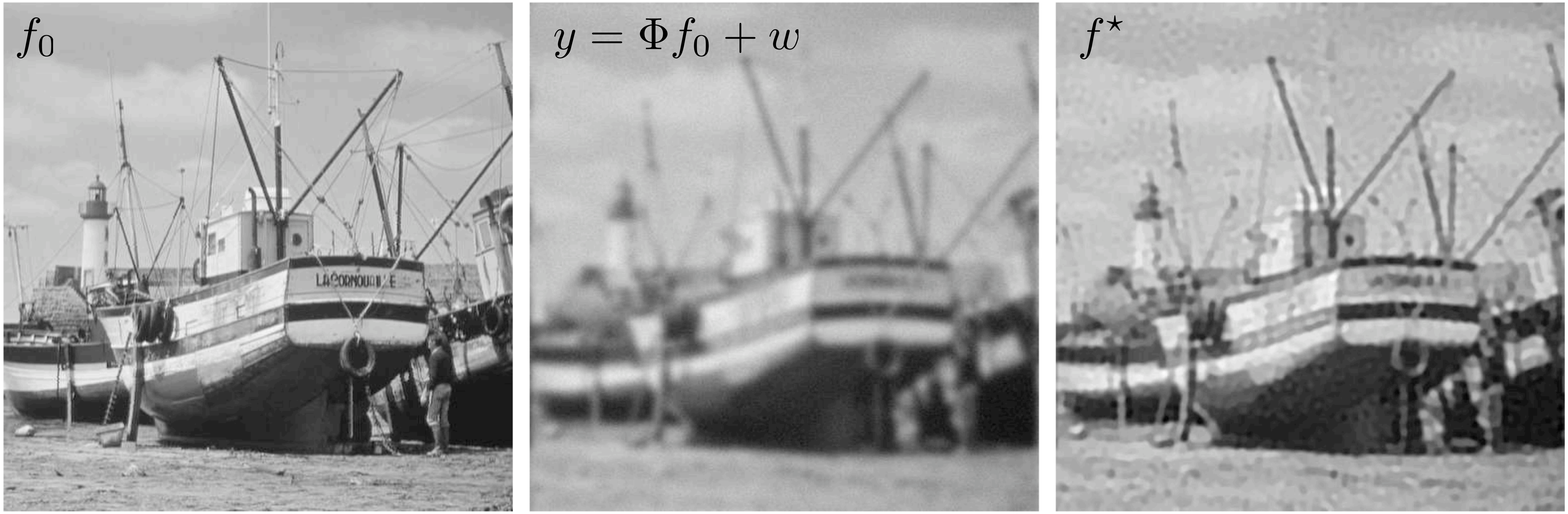
Operator Φ: 70% missing pixels.

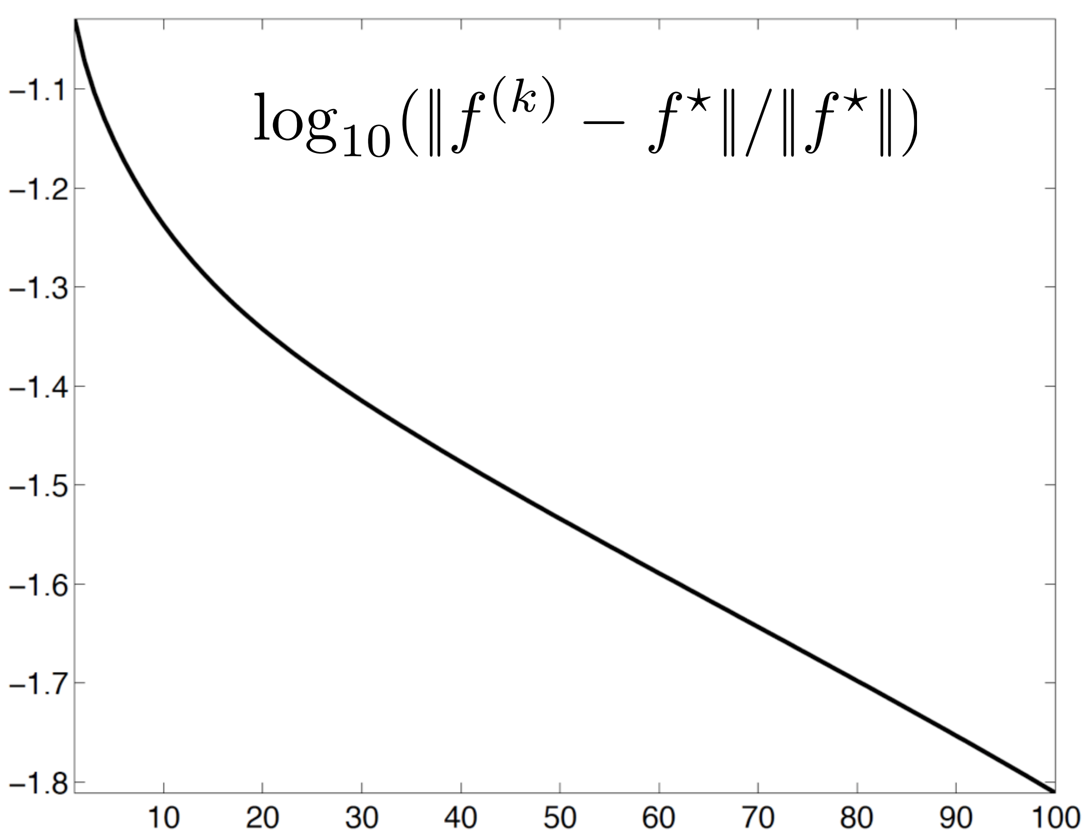
Gaussian noise: $\|w\| = 0.02\|f_0\|_\infty$.

TV constraint: $\tau = 0.6\|f_0\|_{\text{TV}}$.

$\longrightarrow$ Roughly $1/\ell$ convergence speed.

Deconvolution





Operator Φ: Gaussian filter, width 4 pixels.

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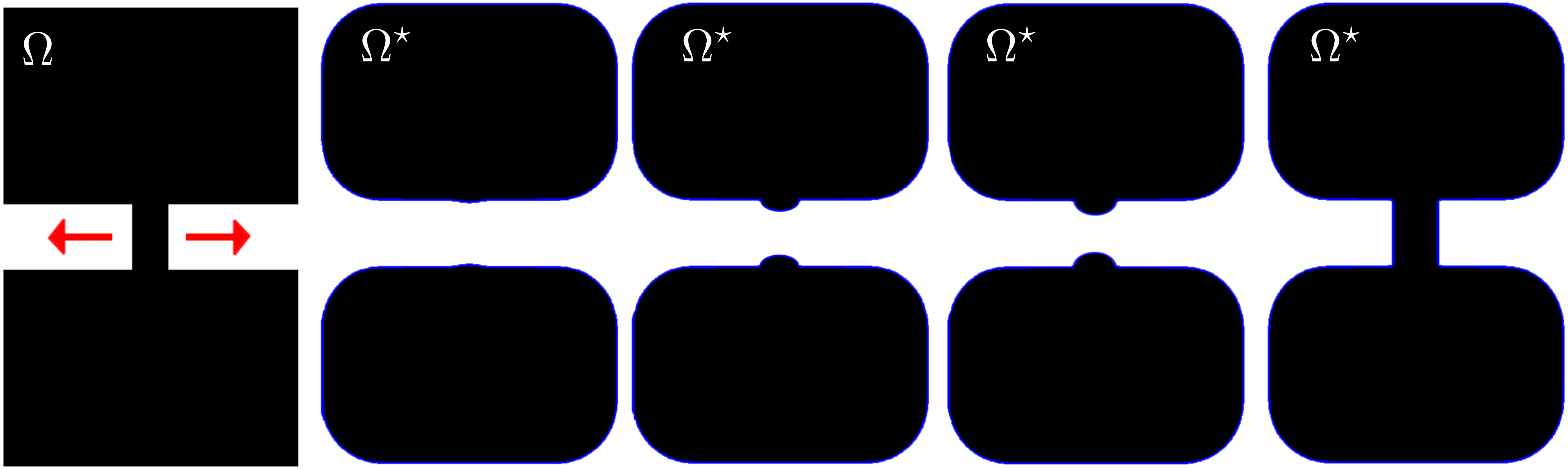
Cheeger Set

Cheeger set: $\Omega^\star \subset \Omega$ solution of $$h(\Omega) = \min_{A \subset \bar{\Omega}} \frac{\text{Perimeter}(A)}{\text{Area}(A)}$$

$\rightarrow$ not necessarily unique.



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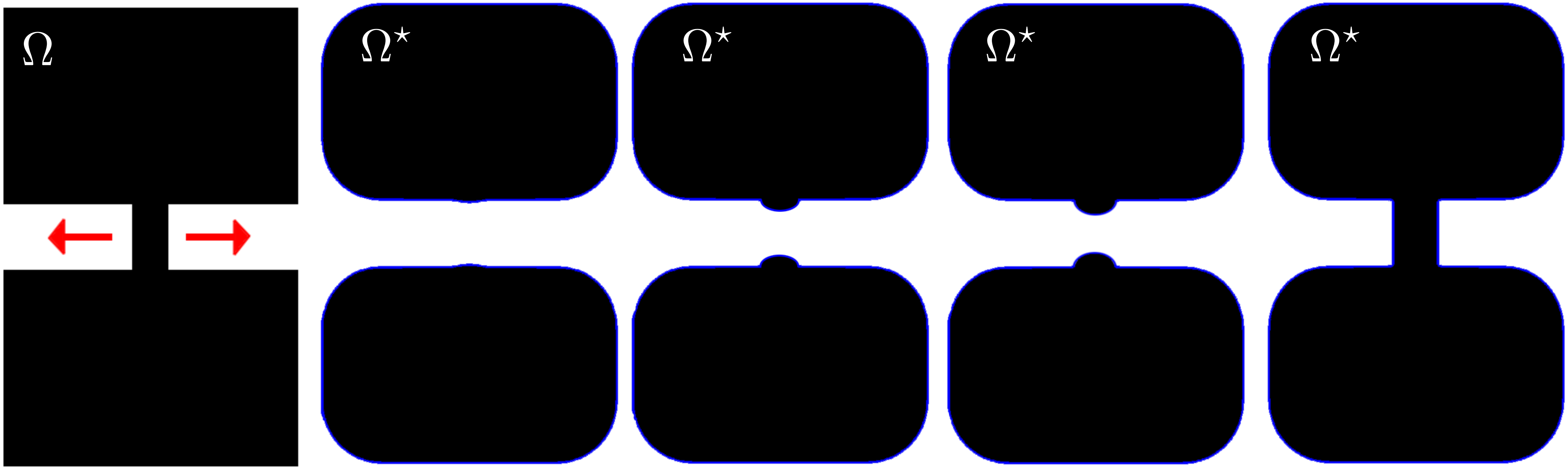
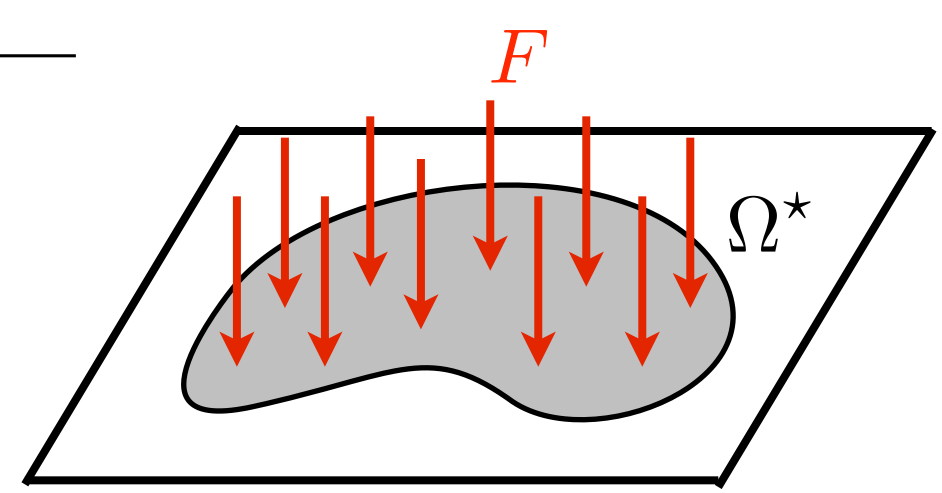


Weighted Cheeger set: $$h(\Omega) = \min_{A \subset \Omega} \frac{\int_{\partial^* A} g d\mathcal{H}^{d-1}}{\int_A F}$$



Landslide modeling: anti-plane flow.

F: external forces (vertical)

g: yeld limit of the soil.

Safety condition: $\mu(\Omega) \geqslant 1$.

Approximation by Projection

Constraint formulation in BV: $$h(\Omega)^{-1} = \sup\left\{\int_\Omega Fu \,:\, u \in K\right\}$$

$$K = \left\{u \in \mathrm{BV}(\Omega) \,:\, \int_\Omega g d|Du| + \int_{\partial\Omega} g|u| d\mathcal{H}^{d-1} \leqslant 1\right\}$$

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Strictly convex relaxation: $$\int_\Omega Fu \longrightarrow \int_\Omega Fu - \frac{\varepsilon}{2}\|u\|^2 \sim -\|u - F/\varepsilon\|^2$$

$$u_\varepsilon = \mathrm{Proj}_K(F/\varepsilon)$$

Theorem: [Buttazo, Carlier, Comte, 2007] $u_\varepsilon \to \alpha\chi_{\Omega^\star}$ in $L^1(\Omega)$

$\Omega^\star$: unique maximal Cheeger set of Ω.

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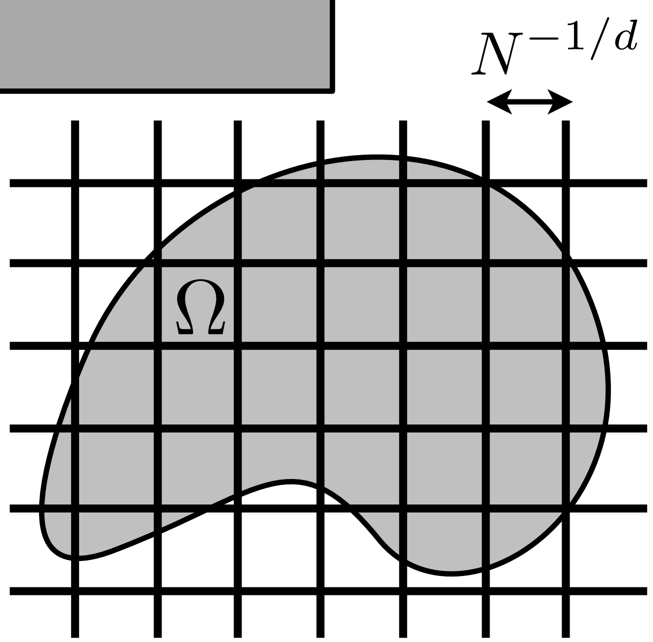
Discretization: grid with N points. $u_\varepsilon^{(N)} = \mathrm{Proj}_{K_N}(F/\varepsilon)$

$$K_N = \left\{ u \in \mathbb{R}^N \;:\; \|u\|_{\mathrm{TV}} \leqslant 1;\ \forall\, n \notin \Omega, u[n] = 0 \right\}$$

$\rightarrow$ Computation of $u_\varepsilon^{(N)}$ using Nesterov on the dual.



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Constraint formulation in BV: $h(\Omega)^{-1} = \sup \left\{ \int_\Omega F u \ : \ u \in K \right\}$

$$K = \left\{ u \in \mathrm{BV}(\Omega) \ : \ \int_\Omega g d|Du| + \int_{\partial\Omega} g|u| d\mathcal{H}^{d-1} \leqslant 1 \right\}$$

Strictly convex relaxation: $\int_\Omega F u \longrightarrow \int_\Omega F u - \frac{\varepsilon}{2}\|u\|^2 \sim -\|u - F/\varepsilon\|^2$

$$u_\varepsilon = \mathrm{Proj}_K(F/\varepsilon)$$

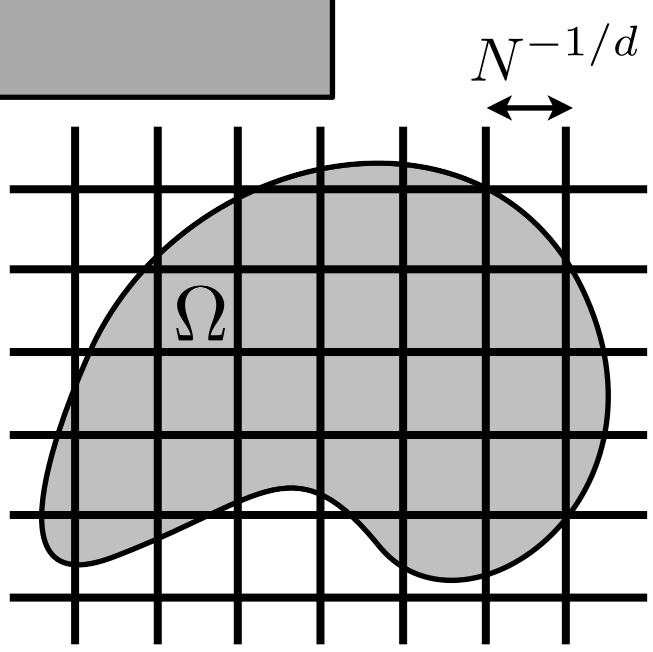
Theorem: [Buttazo, Carlier, Comte, 2007] $u_\varepsilon \rightarrow \alpha \chi_{\Omega^\star}$ in $L^1(\Omega)$

$\Omega^\star$: unique maximal Cheeger set of Ω.

Discretization: grid with N points. $u_\varepsilon^{(N)} = \mathrm{Proj}_{K_N}(F/\varepsilon)$

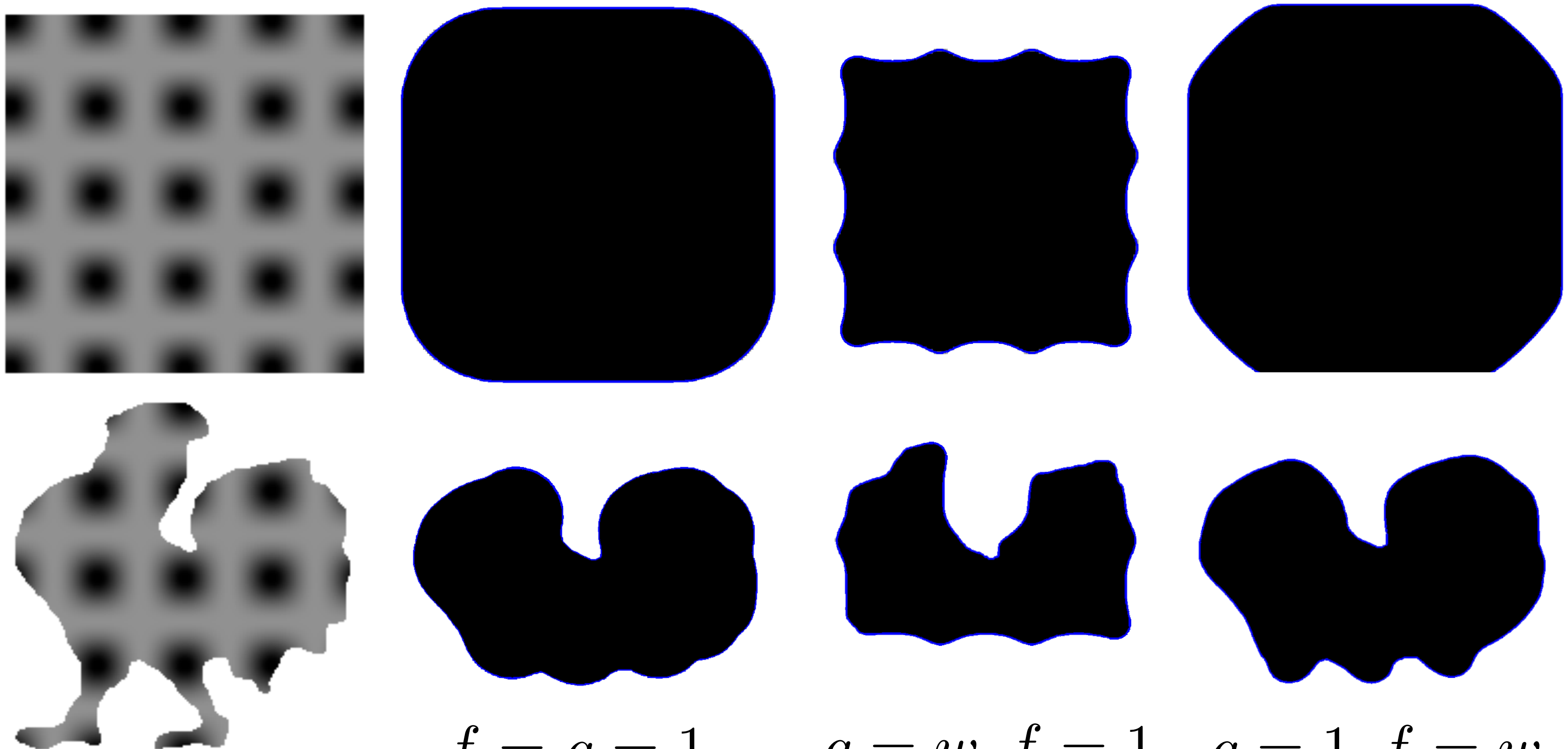
$$K_N = \{ u \in \mathbb{R}^N \ : \ \|u\|_{\mathrm{TV}} \leqslant 1; \ \forall\, n \notin \Omega, u[n] = 0 \}$$

$\rightarrow$ Computation of $u_\varepsilon^{(N)}$ using Nesterov on the dual.



Theorem: [Carlier, Comte, Peyré 2009] $u_\varepsilon^{(N)} \overset{N \rightarrow +\infty}{\longrightarrow} u_\varepsilon$ in $L^2(\Omega)$.

Influence of the Weights

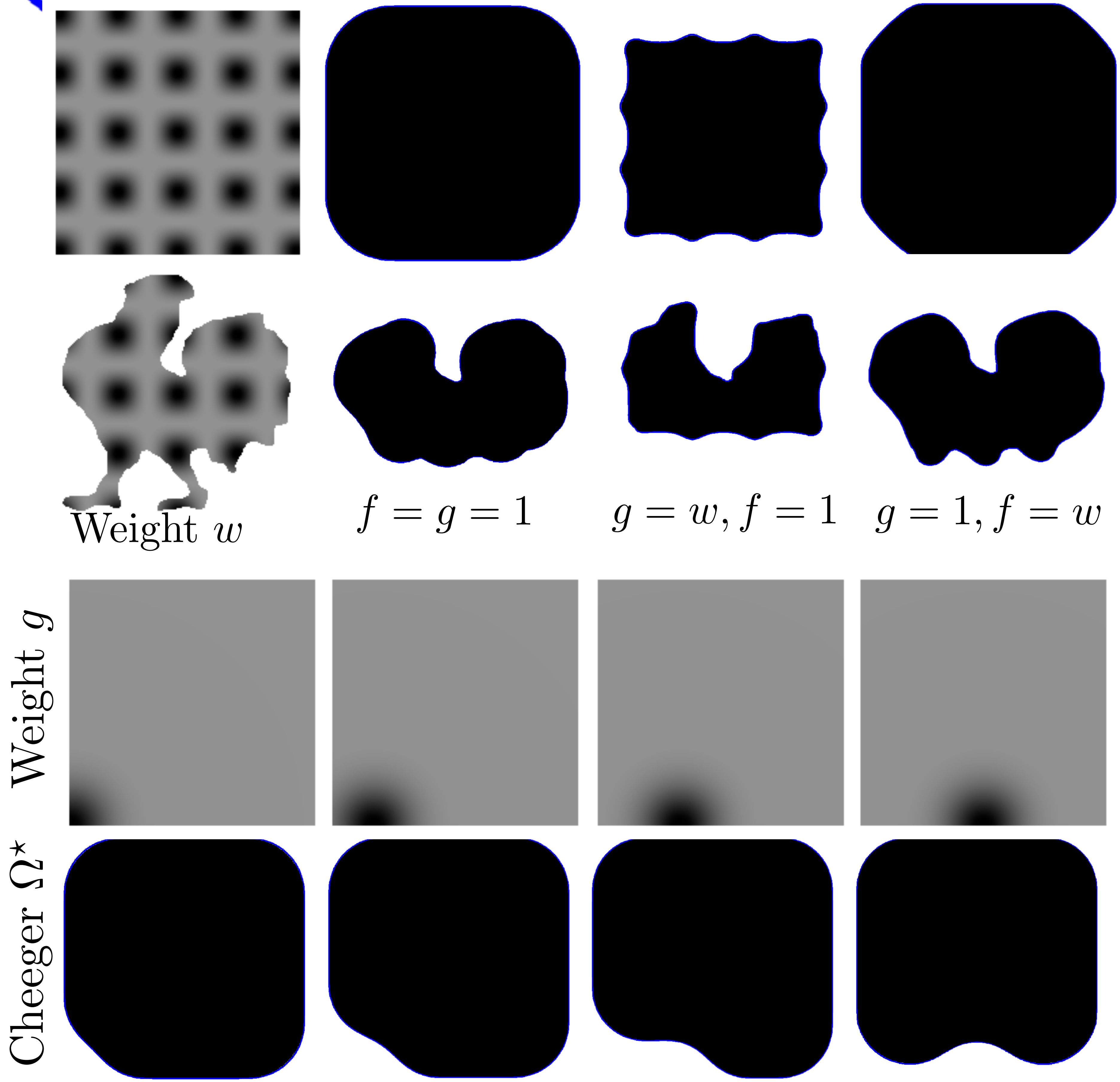


Weight w | $f = g = 1$ | $g = w, f = 1$ | $g = 1, f = w$

$\varepsilon = 10^{-3}$

$N = 256$

Influence of the Weights



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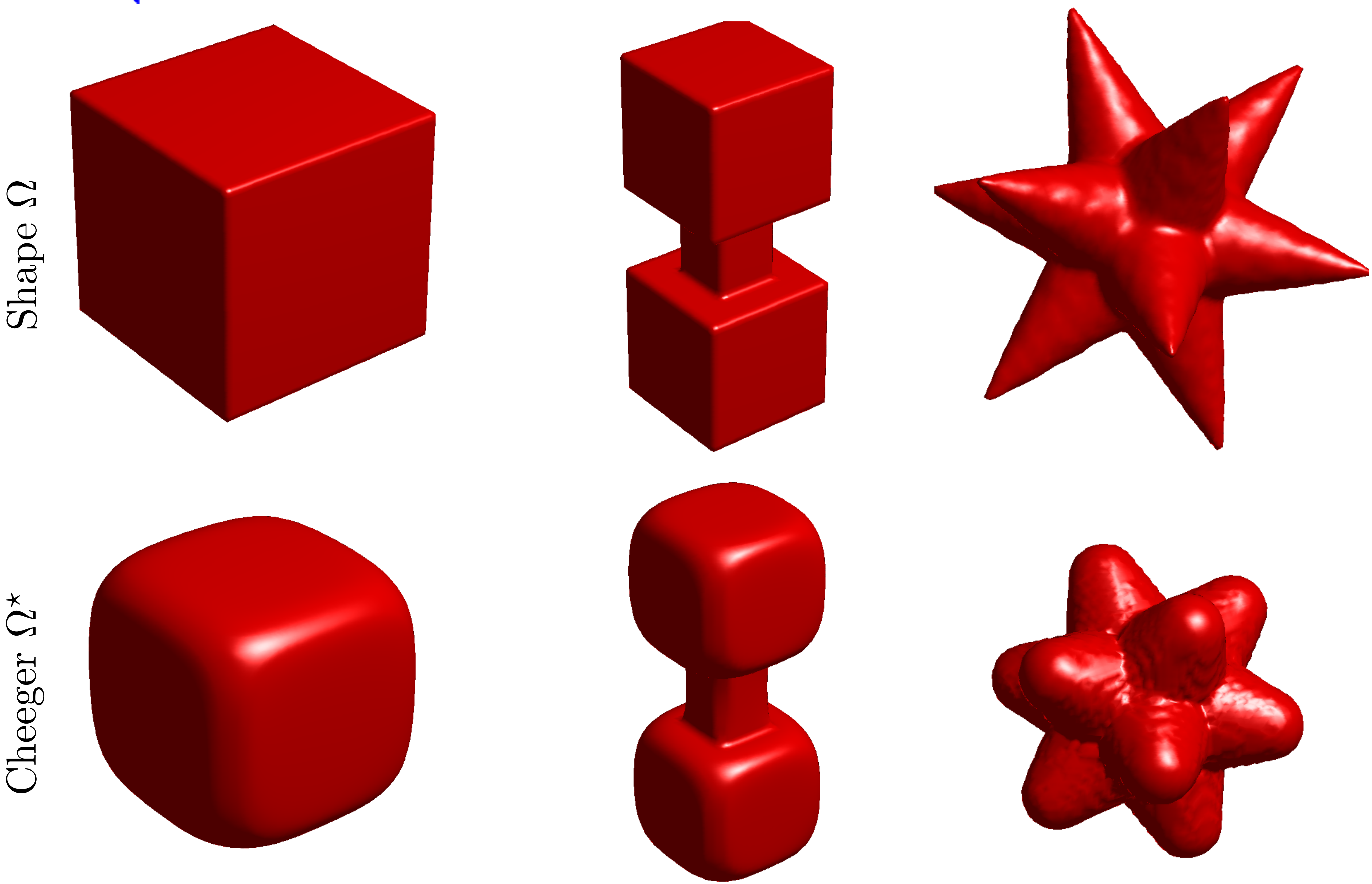
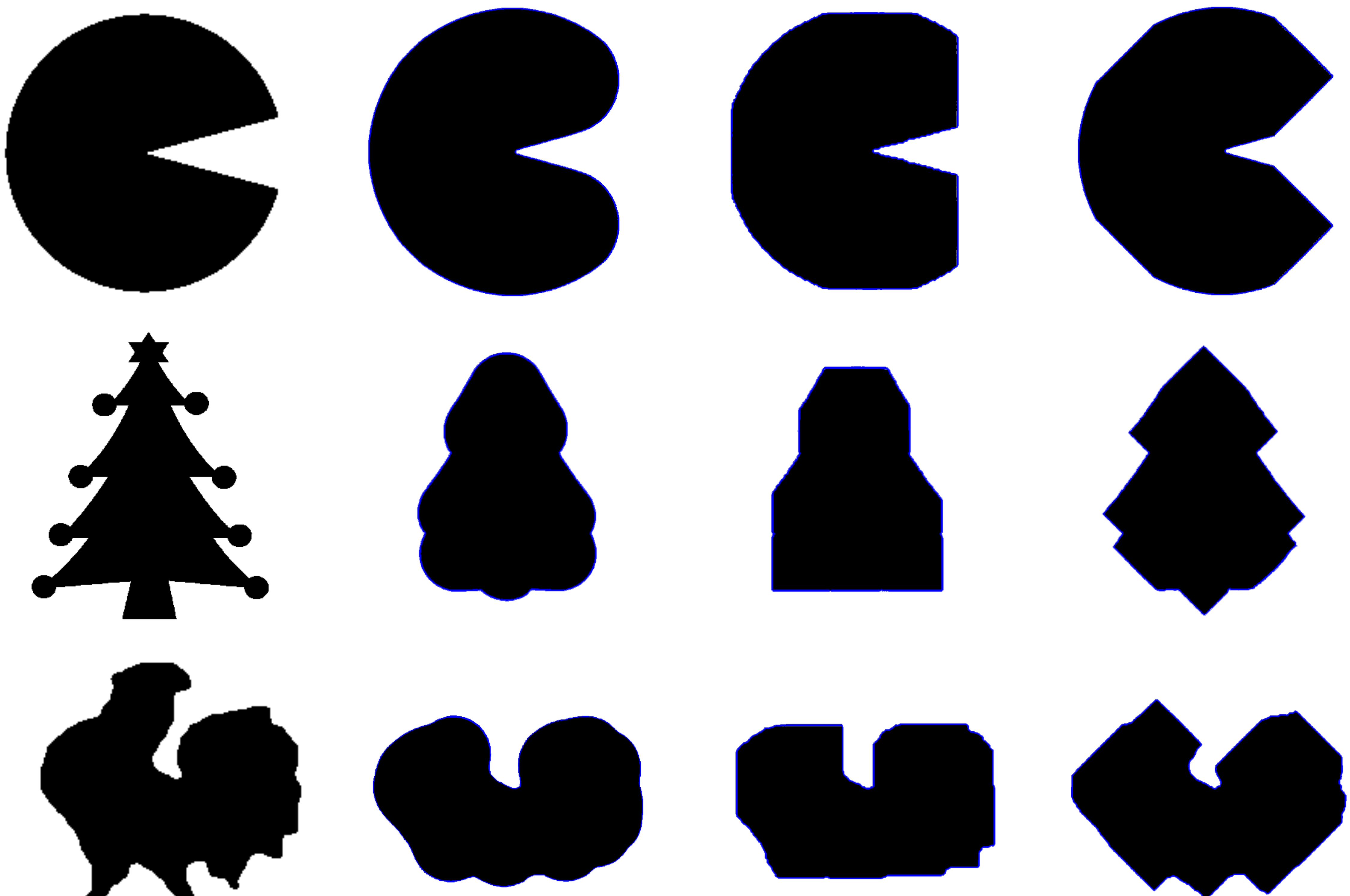
3D Cheeger Sets



Anisotropic Metric

Replace $\int \|\nabla u\|$ by $\int \varphi(\nabla u)$.



Shape Ω | $\varphi(x) = \|x\|$ L^2 Cheeger | $\varphi(x) = |x_1| + |x_2|$ L^1 Cheeger | $\varphi(x) = \max(|x_1|, |x_2|)$ L$^\infty$ Cheeger

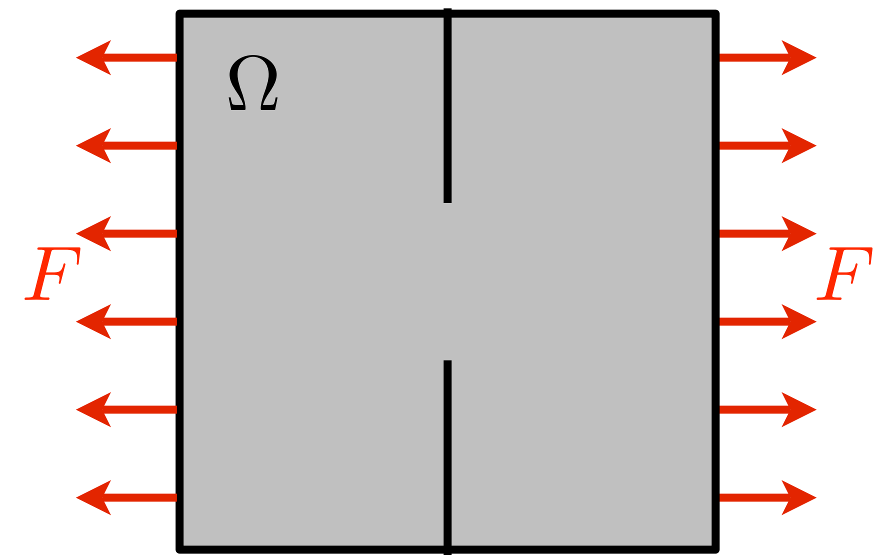
Limit Load Analysis

Incompressible deformation field: $u : \Omega \to \mathbb{R}^2$ with $\boxed{\mathrm{div}(u) = 0}$

Bounded deformation model: $\|u\|_{\mathrm{BD}} = \int_\Omega \|Du\|$ $\qquad Dv = \frac{1}{2}(\nabla u + \nabla^* u)$

$\longrightarrow$ generalize to discontinuous fields v.

Load on the boundary: $L(u) = \int_{\partial\Omega} \langle u,\, F \rangle$



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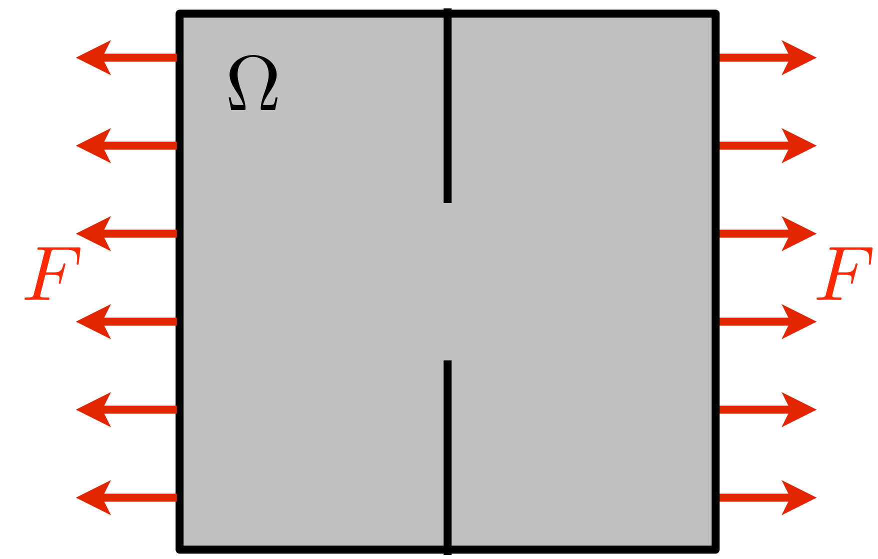

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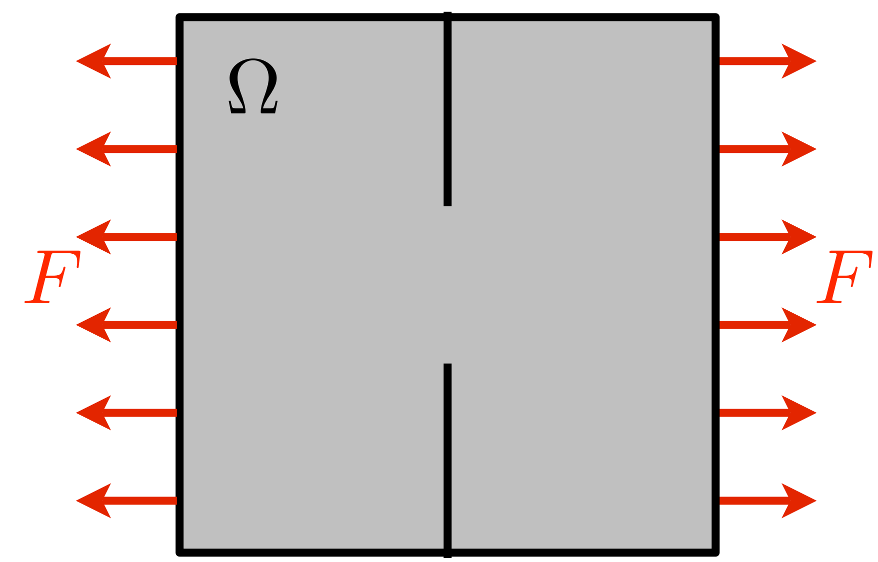

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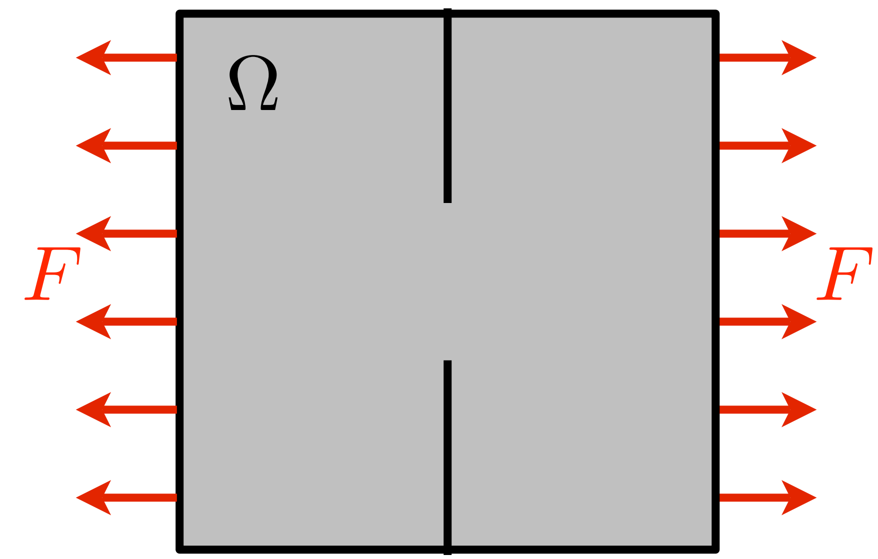


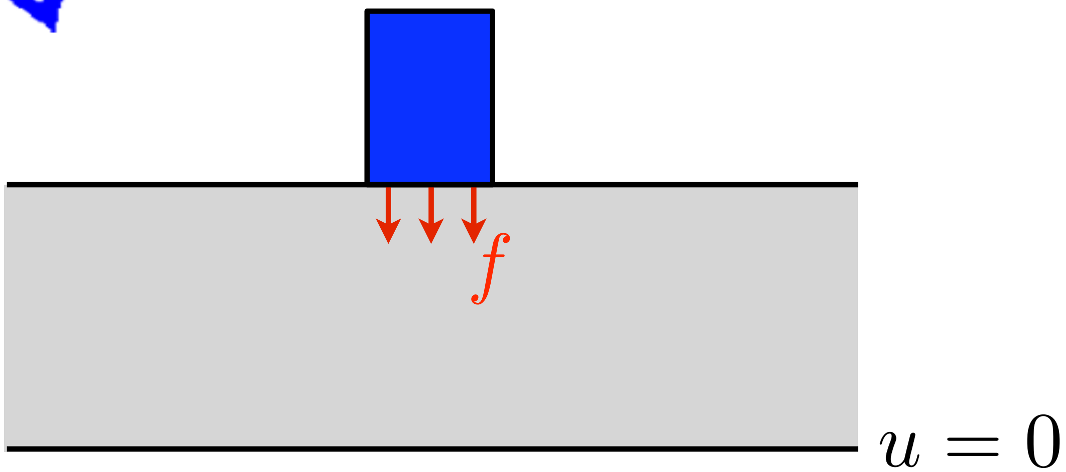
Strictly convex relaxation: $u_\varepsilon = \mathrm{Proj}_K(F/\varepsilon)$

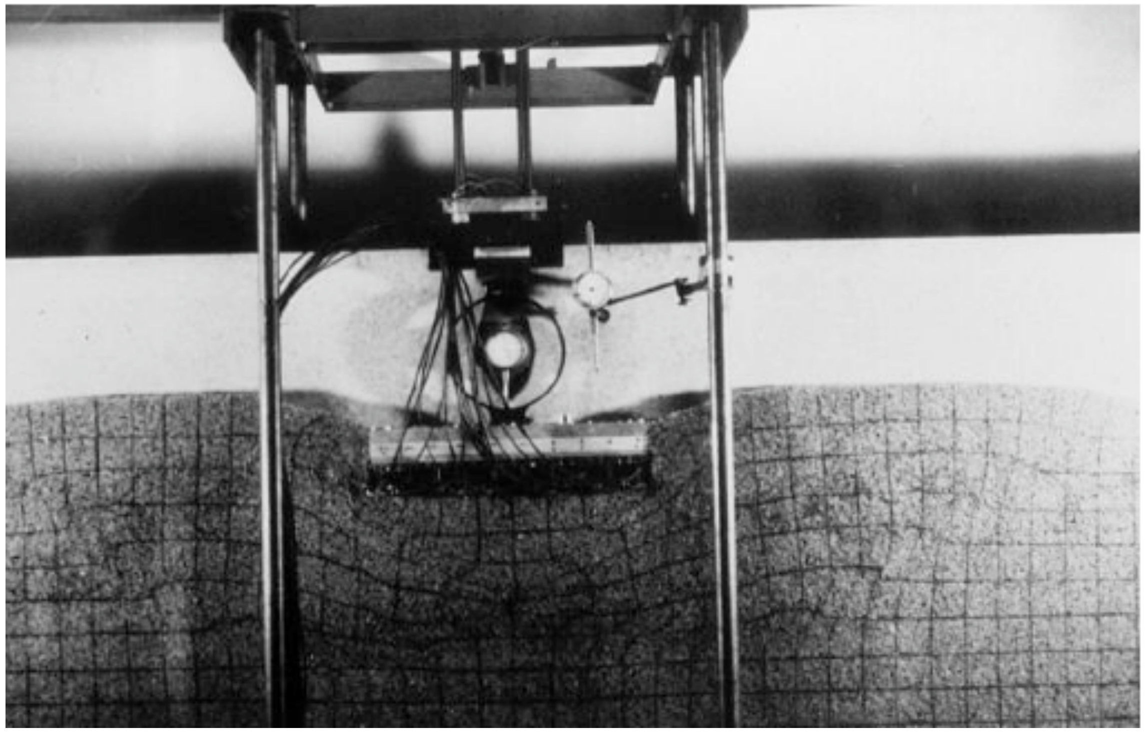
Theorem: [Carlier, Comte, Ionescu, Peyré 2009]

$$u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^\star = \underset{u \in \mathcal{S}}{\operatorname{argmin}} \|u\| \quad \text{in } \mathrm{L}^2(\Omega) \text{ where } \quad \mathcal{S} = \underset{v \in K}{\operatorname{argmax}} \int_{\partial\Omega} \langle u,\, f \rangle$$

Discretization: finite difference $\|u\|_{\mathrm{BD}}$ for $u \in \mathbb{R}^{N \times 2}$.

Numerical Example #1





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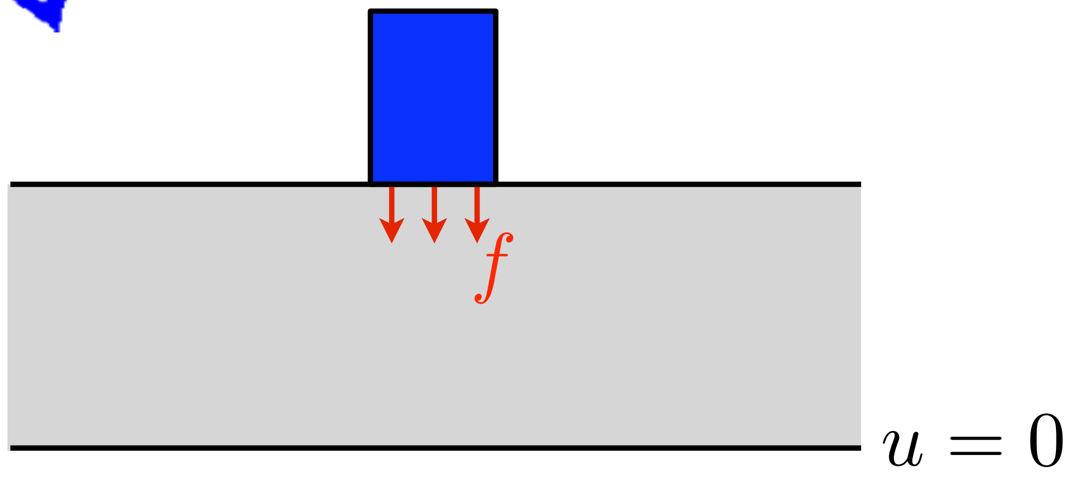
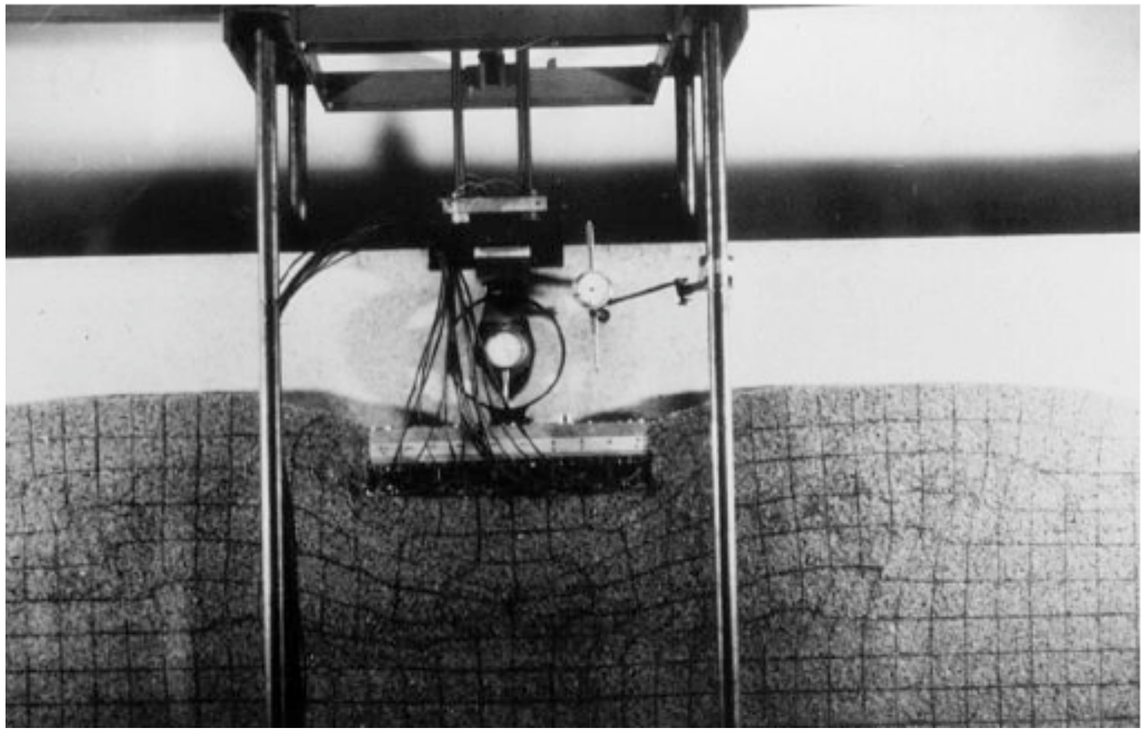
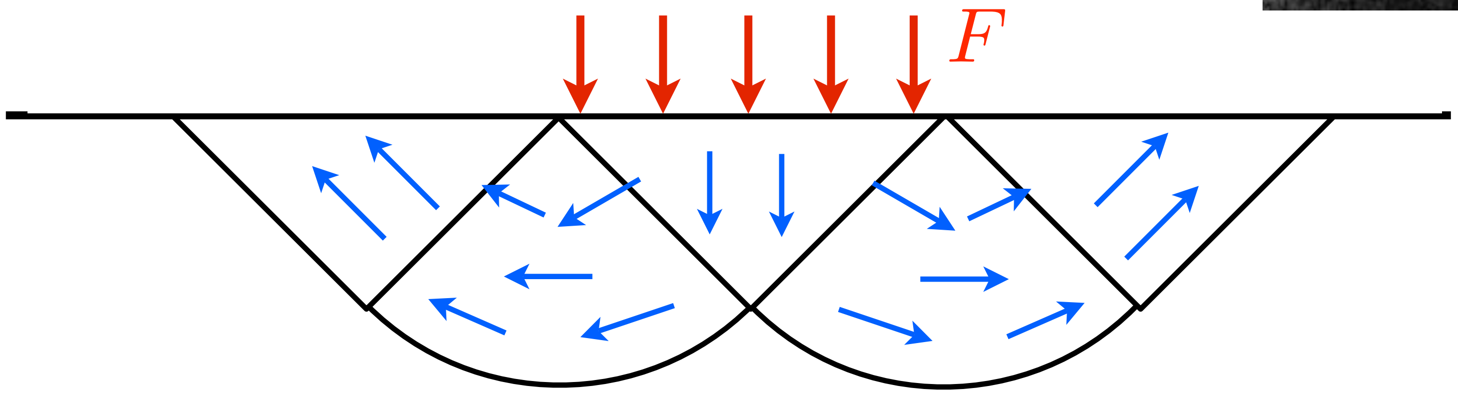




Analytical solution: Prandlt model.



$$h(\Omega)^{-1} = \frac{2+\pi}{\sqrt{2}} \approx 3.64$$

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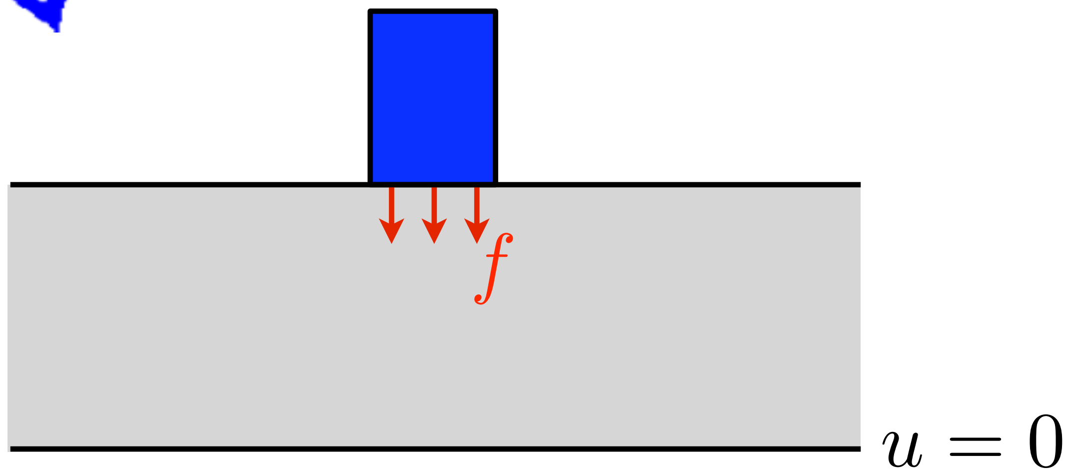

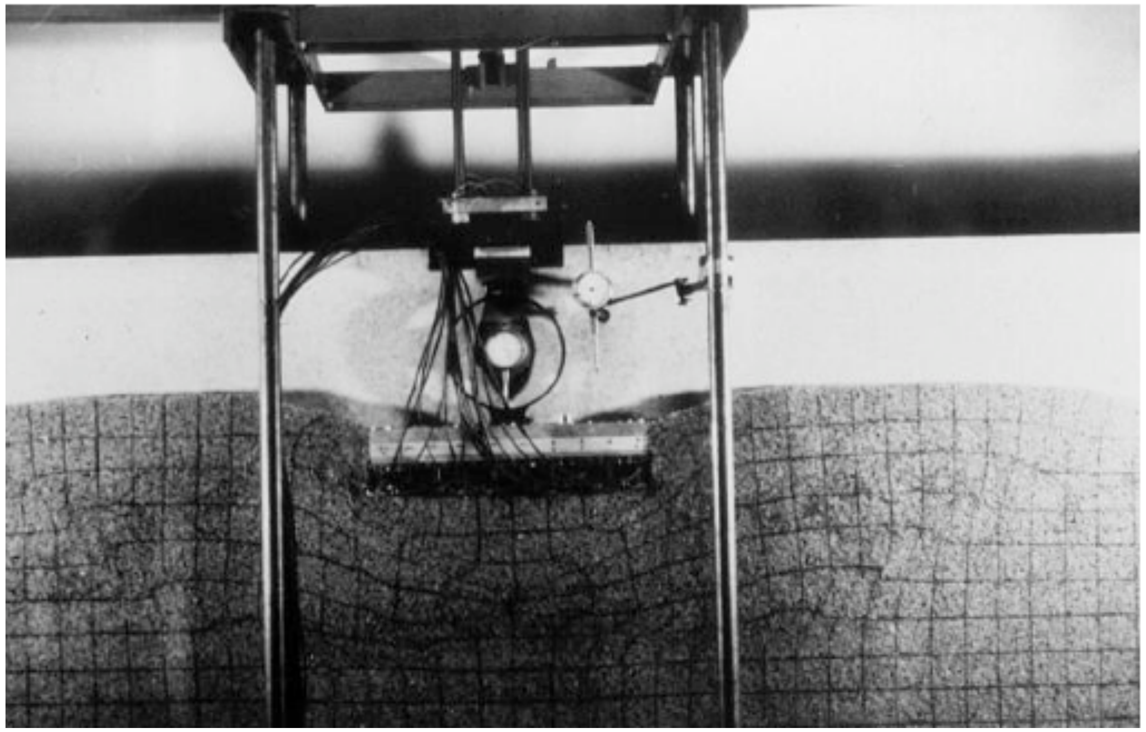

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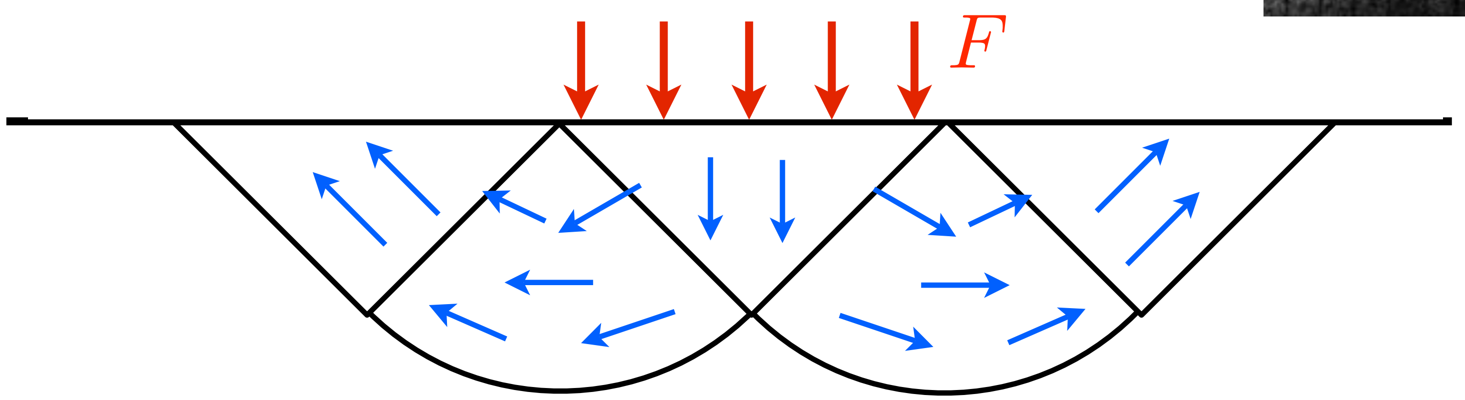
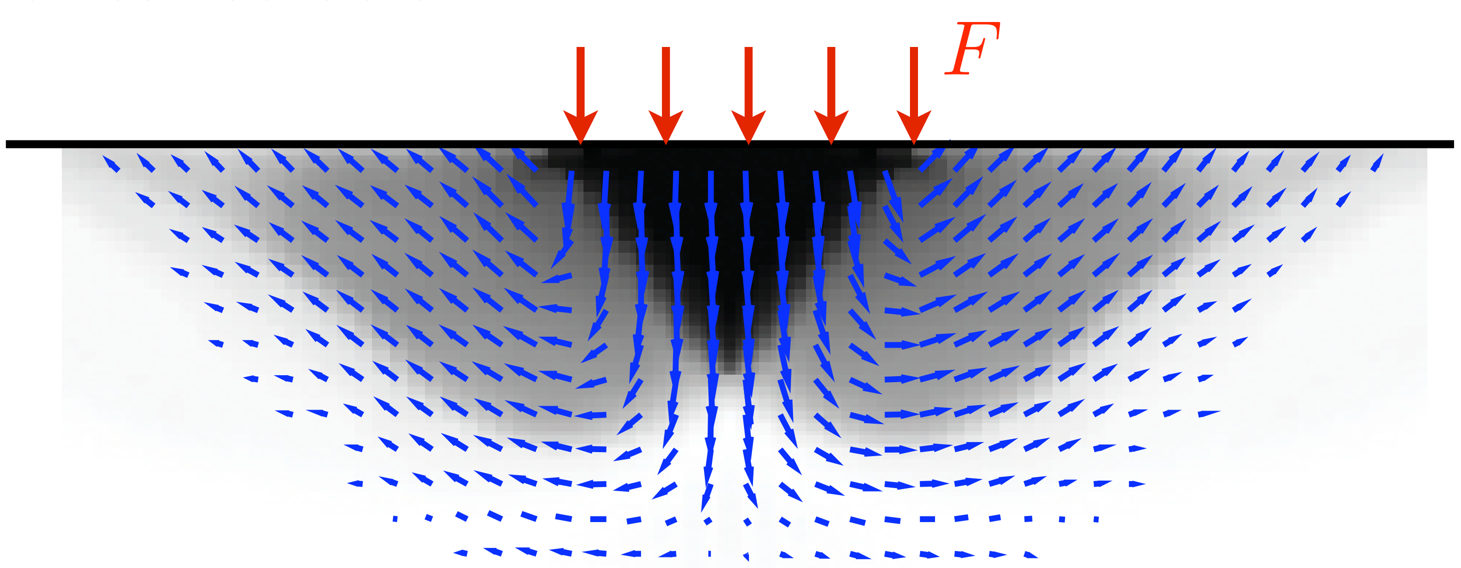


$$h(\Omega)^{-1} = \frac{2+\pi}{\sqrt{2}} \approx 3.64$$

Numerical solution:



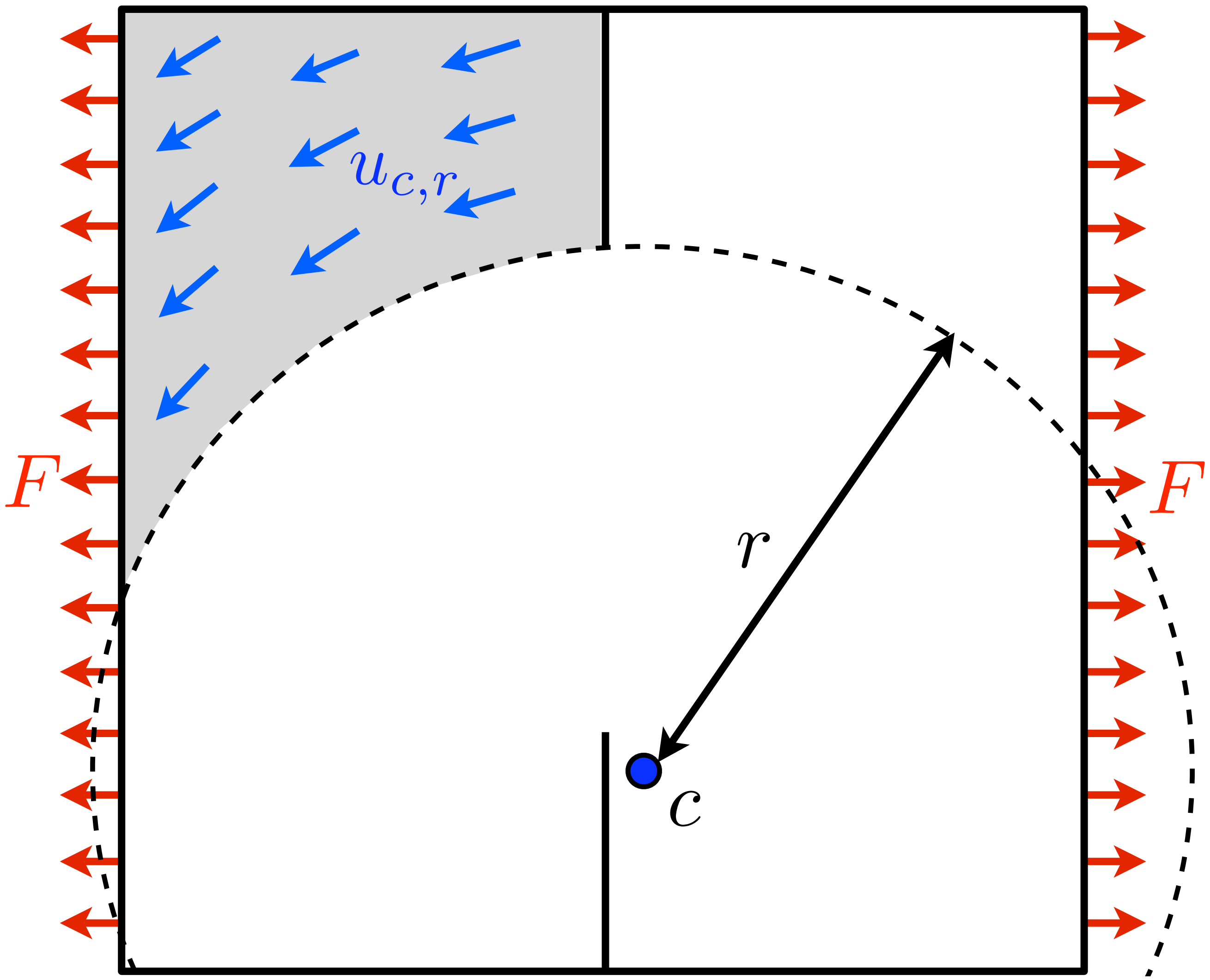
$$\varepsilon = \|P_{\mathrm{div}}(f)\|_{\mathrm{BD}}/100$$

$$N = 100 \times 100$$

$$h_{\varepsilon}^{(N)}(\Omega)^{-1} = 3.85$$

Numerical Example #2

Parameteric family of solutions: $u = u_{c,r}$
[Oudet, Ionescu]



Best parameters $c^\star, r^\star$:

$$h_{c^\star,r^\star}(\Omega)^{-1} = 1.136$$

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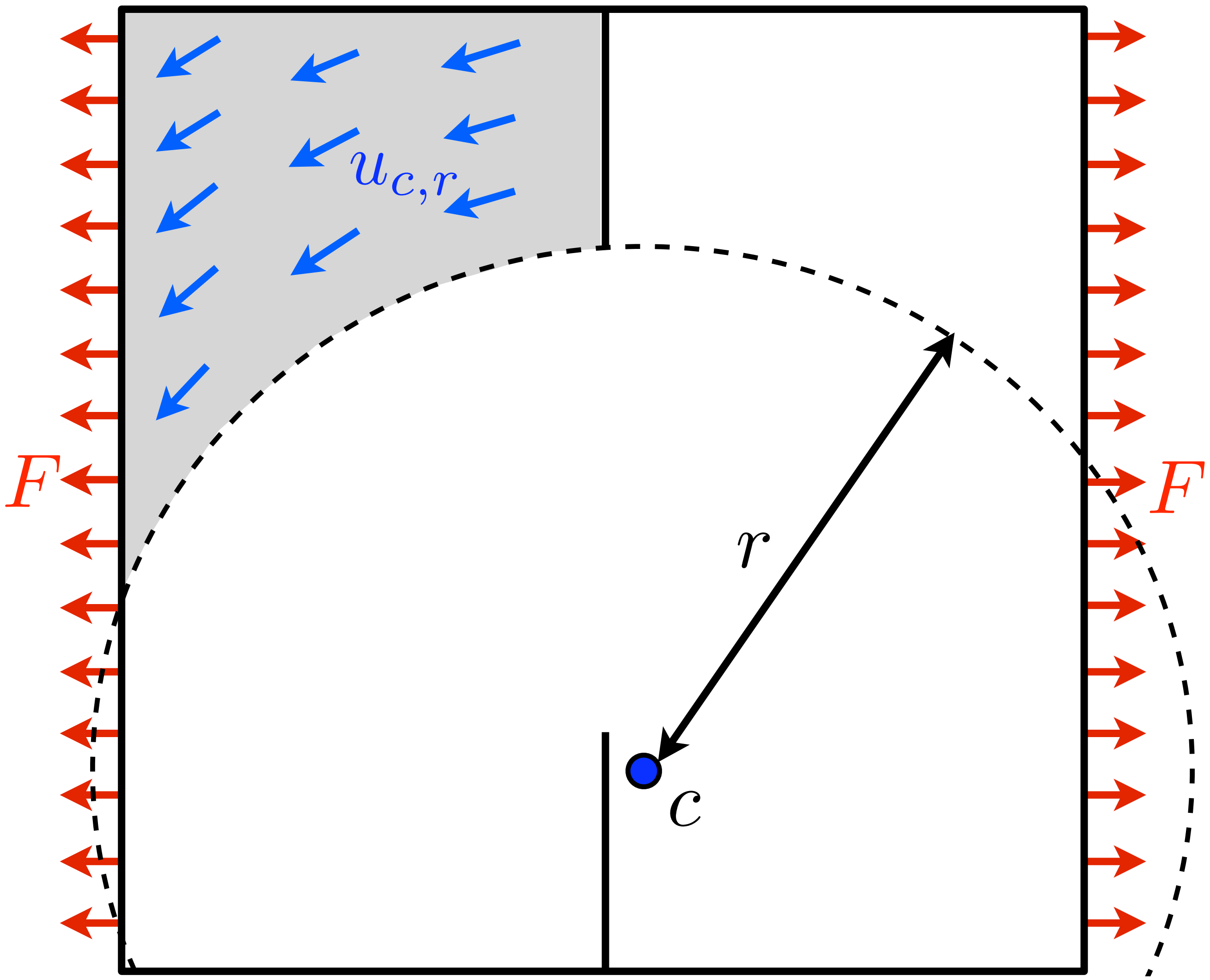

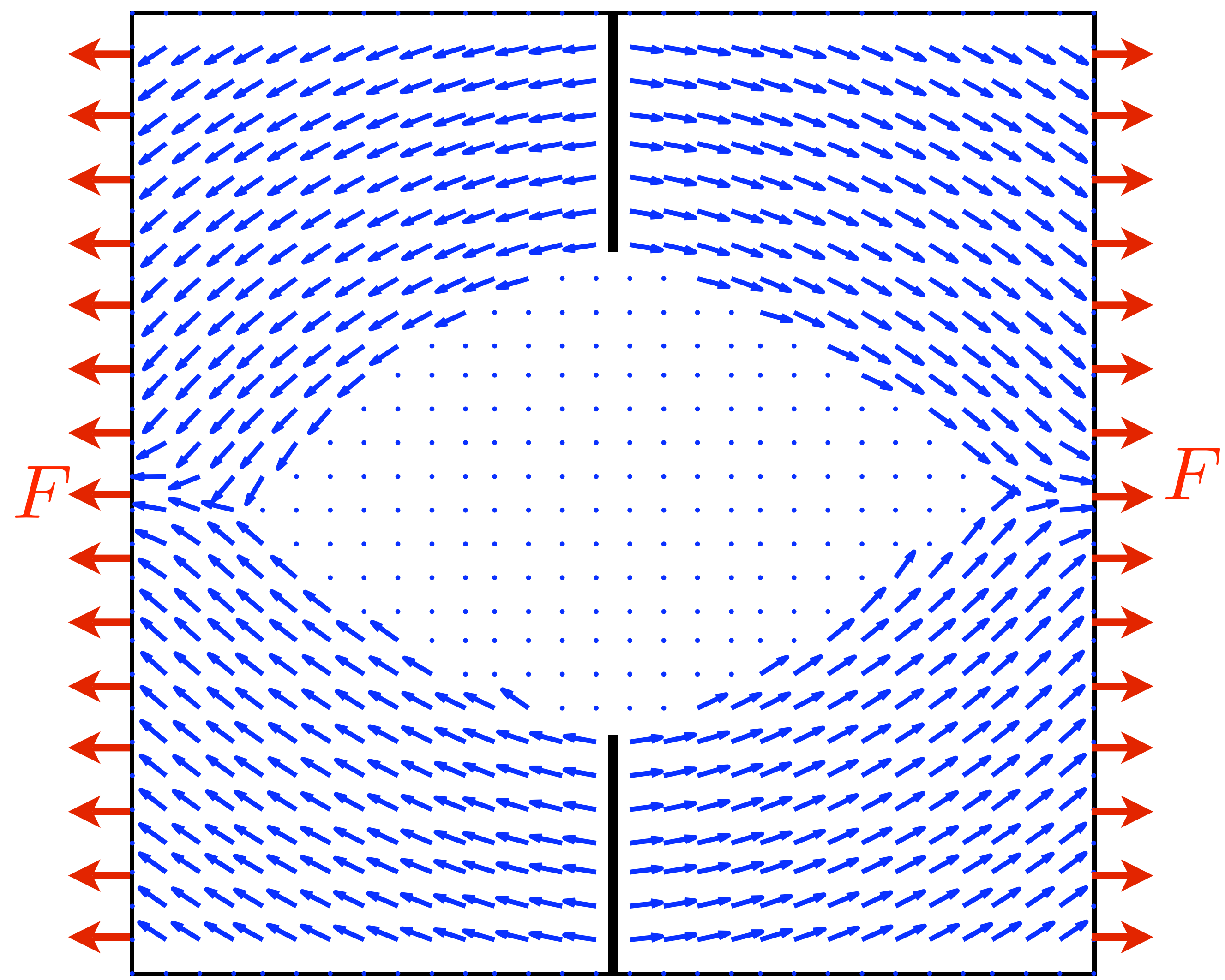

Best parameters $c^\star, r^\star$:

$$h_{c^\star,r^\star}(\Omega)^{-1} = 1.136$$

$$\varepsilon = \|P_{\mathrm{div}}(f)\|_{\mathrm{BD}}/100$$

$$N = 100 \times 100$$

$$h_\varepsilon^{(N)}(\Omega)^{-1} = 1.094$$

Conclusion

- *Dual TV projection:* unconstraint optimization.

$$\min_{\|g\|_{\mathrm{TV}} \leqslant \tau} \|f - g\| \longrightarrow \min_u \frac{1}{2}\|f_0 - \mathrm{div}(u)\|^2 + \tau\|u\|_\infty$$

- *Gradient methods:* Forward-backward or Nesterov (faster).

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$\rightarrow$ projected gradient descent: stability to imperfect projections.

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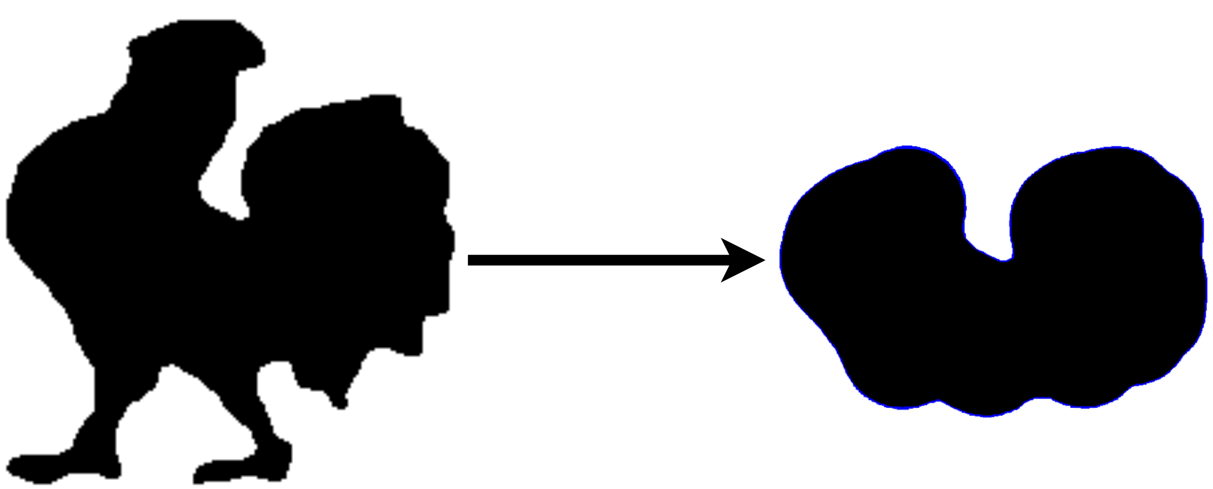
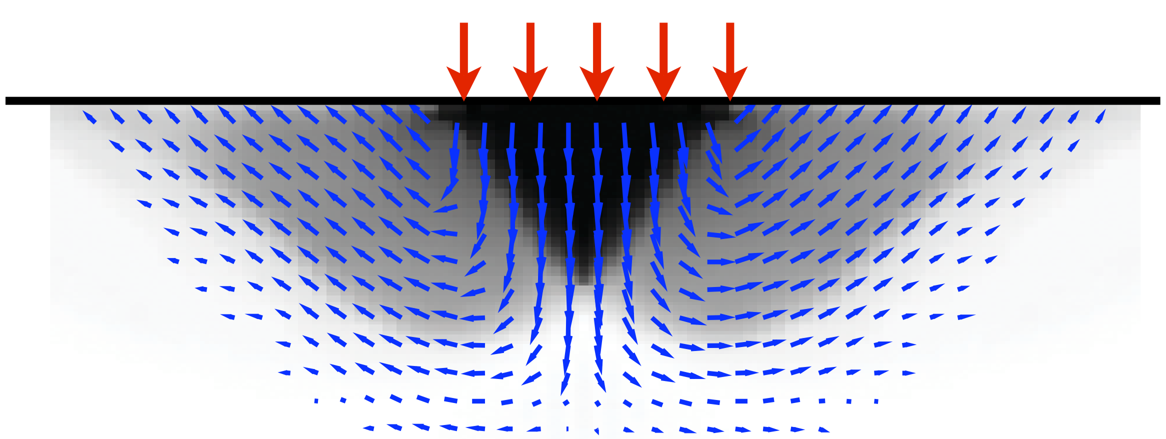
- *Shape optimization:* computing Cheeger sets, plastic mechanics.





Total Variation Texture Synthesis

TV texture synthesis: draw $f \in \mathcal{C}_{\text{TV}} \cap \mathcal{C}_{\text{contrast}}$ at random.

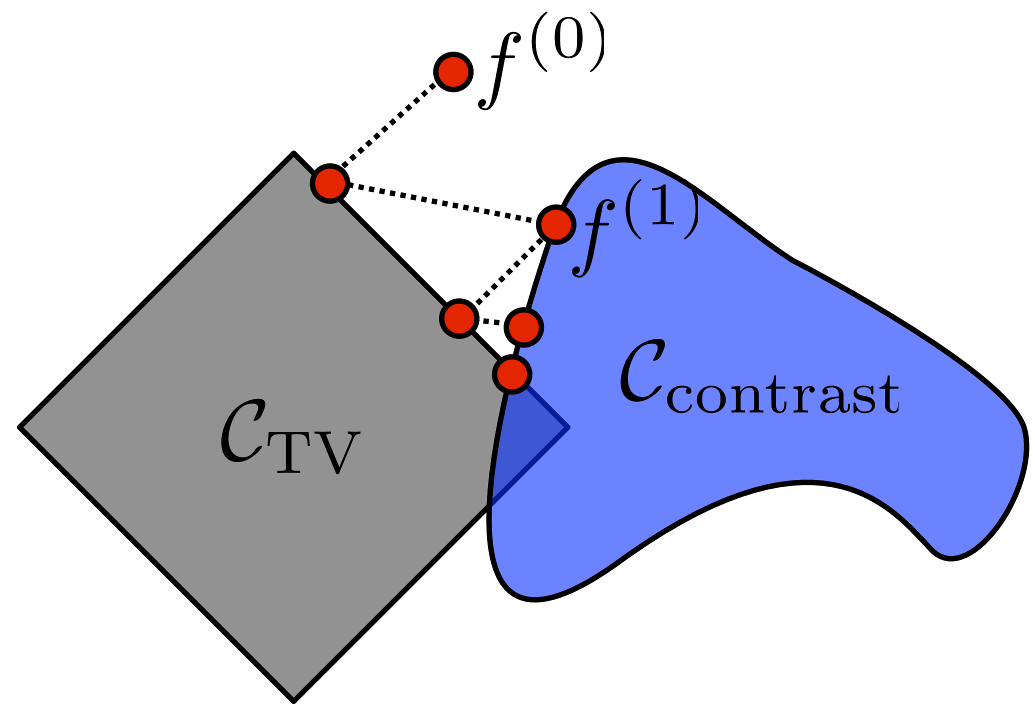
$$\mathcal{C}_{\text{TV}} = \{f \setminus \|f\|_{\text{TV}} \leqslant \tau\} \qquad \mathcal{C}_{\text{contrast}} = \{f \setminus \text{histogram}(f) = h_0\}$$

Iterated projections: $f^{(0)} \leftarrow \text{noise}.$

$$f^{(k+1)} = \text{Equalize}_{h_0}\left(\text{Proj}_{\|\cdot\|_{\text{TV}} \leqslant \tau}(f^{(k)})\right)$$

$\mathcal{C}_{\text{contrast}}$ is non-convex.

Local convergence [Lewis, Malick, 2008]



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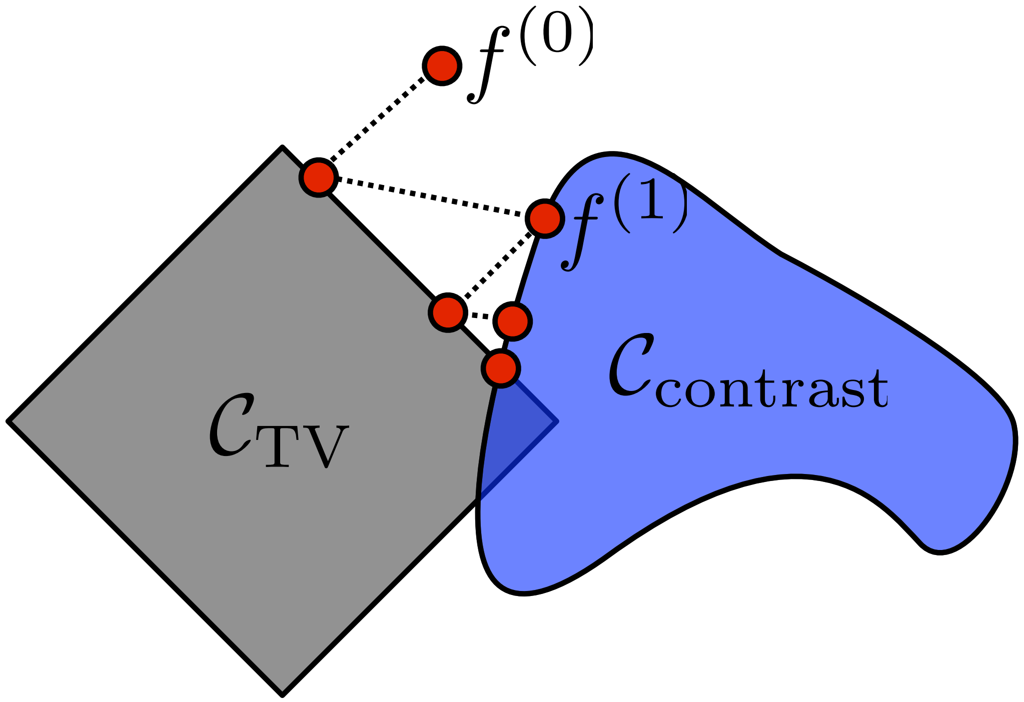
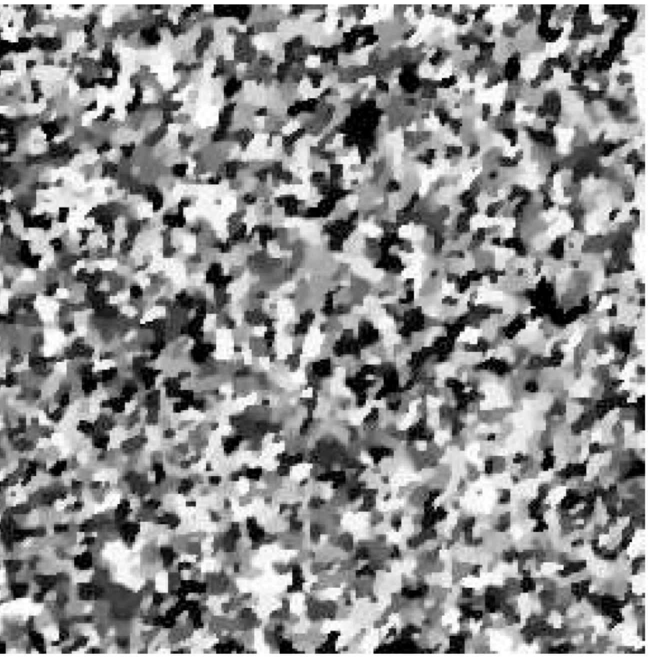
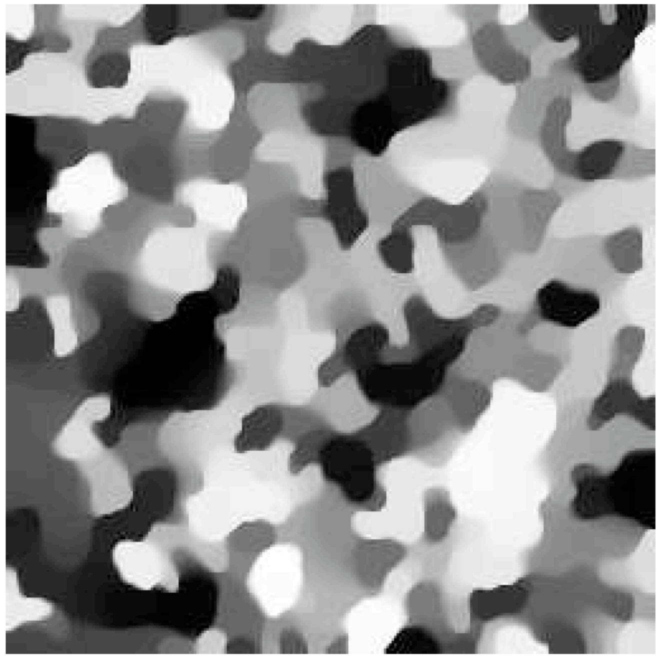
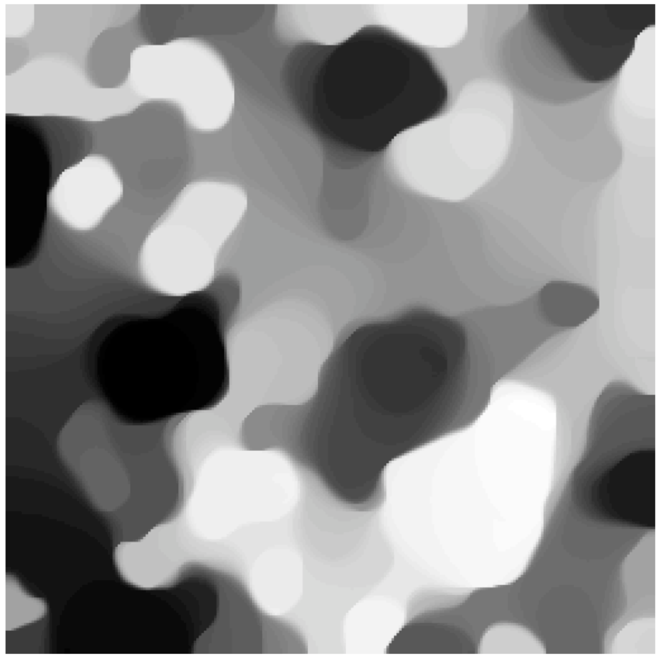



$\|f^{(0)}\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 4$


$\|f^{(0)}\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 8$


$\|f^{(0)}\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 16$


$\|f^{(0)}\|_{\text{TV}}/\|f^\star\|_{\text{TV}} = 32$