On existence of a limit average value for a optimal control problem with horizon tending to infinity

Marc Quincampoix
Université de Brest

Porquerolles, October 2009

M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems,(*submitted*) (2009)

An Optimal Control Problem

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^{t} h(y(s,u,y_0), u(s))ds,$$

where $s \mapsto y(s,u,y_0)$ **denotes the solution to**

$$y'(s) = g(y(s), u(s)), \quad y(0) = y_0.$$

$g : I\!R^d \times U \to I\!R^d$ **Lipschitz,** U **compact,** g h **bounded.**

PROBLEM : Existence of a limit of $V_t(y_0)$ **as** $t \to +\infty$**.**

No ergodicity condition here (Lions-Papanicolaou- Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, ...)The limit may depend on the initial condition

Contents

Introduction

Definition 1 ***The problem*** $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ ***has a*** *limit value* ***if***

$$V(y_0) := \lim_{t\to\infty} V_t(y_0) = \lim_{t\to\infty} \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds.$$

Definition 2 ***The problem*** $\Gamma(y_0)$ ***has a*** *uniform value* ***if it has a limit value*** $V(y_0)$ ***and if:***

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds \leq V(y_0) + \varepsilon.$$

Examples

• **Example 1: here** $y \in I\!R^2$ **(seen as the complex plane** $i^2 = -1$**), there is no control**

$$y'(t) = i\, y(t),$$

$$V_t(y_0) \xrightarrow[t \to \infty]{} \frac{1}{2\pi |y_0|} \int_{|z|=|y_0|} h(z) dz,$$

and since there is no control, the value is uniform.

• **Example 2: in the complex plane again, but now** $g(y,u) = i\ y\ u$**, where** $u \in U$ **a given bounded subset of** $I\!R$**, and** h **is continuous in** y**.**

• **Example 3:** $g(y,u) = -y+u$, **where** $u \in U$ **a given bounded subset of** IR^d, **and** h **is continuous in** y**.**

• **Example 4: in** IR^2**. The initial state is** $y_0 = (0,0)$ **and** $U = [0,1]$, **and the cost is** $h(y) = 1 - y_1(1-y_2)$**.**

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1-y_1(s)) \\ u^2(s)(1-y_1(s)) \end{pmatrix}.$$

One can easily observe that the reachable set $G(y_0) \subset [0,1]^2$**.**

If $u = \varepsilon > 0$ **constant,** $y_1(t) = 1 - \exp(-\varepsilon t)$ **and** $y_2(t) = \varepsilon y_1(t)$**.**
So we have $V_t(y_0) \xrightarrow[t\to\infty]{} 0$. **Existence of a Uniform Value**

No ergodicity :

$$\{y \in [0,1]^2, \lim_{t\to\infty} V_t(y) = \lim_{t\to\infty} V_t(y_0)\} = [0,1] \times \{0\},$$

and starting from y_0 **it is possible to reach no point in** $(0,1] \times \{0\}$**.**

Examples

• **Example 5: in** IR^2, $y_0 = (0,0)$, **control set** $U = [0,1]$, $y'(t) = (y_2(t), u(t))$, **and** $h(y_1, y_2) = 0$ **if** $y_1 \in [1,2]$, $= 1$ **otherwise.**

We have $u(s) = y_2'(s) = y_1''(s)$,

Interpretation: u **"acceleration"**, y_2 **"speed"**, y_1 **the "position".**

If $u = \varepsilon$ **constant, then** $y_2(t) = \sqrt{2\varepsilon y_1(t)}$ $\forall t \geq 0$.

Limit Value: $V_T(y_0) \xrightarrow[T\to\infty]{} 1/2$

No Uniform Value.

Assumptions and Notations

$$\begin{cases} \textbf{The function } h : I\!R^d \times U \longrightarrow I\!R \textbf{ is measurable and bounded} \\ \exists L \geq 0, \forall (y, y') \in I\!R^{2d}, \forall u \in U,\ \|g(y,u) - g(y',u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y,u) \in I\!R^d \times U,\ \|g(y,u)\| \leq a(1 + \|y\|) \end{cases}$$

Average cost induced by u between 0 and t by:

$$\gamma_t(y_0, u) := \frac{1}{t}\int_0^t h(y(s,u,y_0), u(s))ds$$

The corresponding Value function satisfies $V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u)$

$$\textbf{for } m \geq 0\textbf{,}\ \gamma_{m,t}(y_0,u) := \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0), u(s))ds,$$

Contents

A Technical Lemma

We define $V^-(y_0) := \liminf_{t\to+\infty} V_t(y_0)$**,** $V^+(y_0) := \limsup_{t\to+\infty} V_t(y_0$

Lemma 3 ***For every*** m_0 ***in*** IR_+***, we have:***

$$\sup_{t>0}\inf_{m\le m_0} V_{m,t}(y_0) \ge V^+(y_0) \ge V^-(y_0) \ge \sup_{t>0}\inf_{m\ge 0} V_{m,t}(y_0).$$

Sketch of the Proof: We first prove $\sup_{t>0}\inf_{m\le m_0} V_{m,t}(y_0) \ge V^+(y_0)$**. Suppose by contradiction that** $\exists \varepsilon > 0\ \forall t > 0$ **we have** $\inf_{m\le m_0} V_{m,t}(y_0) \le V^+(y_0) - \varepsilon$ **. Hence for any** $t > 0$ **there exists** $m \le m_0$ **with** $V_{m,t}(y_0) \le V^+(y_0) - (\varepsilon/2)$**. Now observe that**

$$V_{m,t}(y_0) \ge \frac{m_0+t}{t} V_{m_0+t}(y_0) - 2\frac{m_0}{t}.$$

Passing to the limsup as $t \to +\infty$ **we obtain a contradiction.**

We now prove $V^{-}(y_0) \geq \sup_{t>0} \inf_{m \leq 0} V_{m,t}(y_0)$. **Assume on the contrary that it is false. Then there exists** $\varepsilon > 0$ **and** $t > 0$ **such that** $V^{-}(y_0) + \varepsilon \leq \inf_{m \leq 0} V_{m,t}(y_0)$. **So for any** $m \geq 0$**, we have** $V^{-}(y_0) + \varepsilon \leq V_{m,t}(y_0)$. **We will obtain a contradiction by concatenating trajectories.**

Recall the result of the Lemma

$$\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq V^{+}(y_0) \geq V^{-}(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0),$$

and define

Definition 4

$$V^{*}(y_0) = \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

A first result in Nonexpansive case

Denote by $G(y_0) := \{y(t, u, y_0), t \geq 0, u \in \mathcal{U}\}$ **the reachable set**

Theorem 5 ***(H'1)*** $h(y, u) = h(y)$ ***only depends on the state,***
(H'2) $G(y_0)$ ***is bounded,***
(H'3) $\forall (y_1, y_2) \in G(y_0)^2$, $\sup_{u \in U} \inf_{v \in U} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq 0$.
Then $\Gamma(y_0)$ ***has a*** *limit value* $V_t(y_0) \xrightarrow[t \to +\infty]{} V^*(y_0)$***. The convergence of*** $(V_t)_t$ ***to*** V^* ***is uniform over*** $G(y_0)$***, and we have*** $V^*(y_0) = \sup_{t \geq 1} \inf_{m \geq 0} V_{m,t}(y_0) = \inf_{m \geq 0} \sup_{t \geq 1} V_{m,t}(y_0) = \lim_{m \to \infty, t \to \infty} V_{m,t}(y_0)$***. Moreover the value of*** $\Gamma(y_0)$ ***is uniform.***

• **Example 1: here** $y \in I\!R^2$ **(seen as the complex plane** $i^2 = -1$**), there is no control**

$$y'(t) = i\, y(t),$$

• **Example 2: in the complex plane**

$$y'(t) = i\, y(t)\, u(t)$$

, where $u \in U$ **a given bounded subset of** $I\!R$**, and** h **is continuous in** y**.**

• **Example 3:** $g(y, u) = -y + u$**, where** $u \in U$ **a given bounded subset of** $I\!R^d$**, and** h **is continuous in** y**.**

Sketch of the proof of the first result

Lemma 6 $\forall T>0, \forall \varepsilon>0, \forall (y_1,y_2)\in G(y_0)^2, \forall u\in\mathcal{U},\ \exists v\in\mathcal{U},$

$$\forall t\in[0,T],\ \|y(t,u,y_1)-y(t,v,y_2)\|\leq\|y_1-y_2\|+\varepsilon.$$

Proposition 7 $\forall\varepsilon>0, \exists m_0, \sup_{t>0}\inf_{m\leq m_0} V_{m,t}(y_0)\leq\sup_{t>0}\inf_{m\geq 0} V_{m,t}(y_0)+2\varepsilon$

- $(V_T(y_0))_{T>0}$ **is equicontinuous (Lemma 6 +continuity of** h**)**
- **Define** $G^m(y_0):=\{y(t,u,y_0), t\leq m, u\in\mathcal{U}\}$ **the reachable set in time** m**.**

$$\forall\varepsilon, \exists m_0, \forall z\in G(y_0), \exists z'\in G^{m_0}(y_0)\ \textbf{ such that }\ \|z-z'\|\leq\varepsilon.$$

- **We have** $\inf_{m\geq 0} V_{m,t}(y_0)=\inf\{V_t(z), z\in G(y_0)\}$**, and** $\inf_{m\leq m_0} V_{m,t}(y_0$ $\inf\{V_t(z), z\in G^{m_0}(y_0)\}$**. By steps 1 and 2** $\inf\{V_t(z), z\in G^{m_0}(y_0)\}\leq \inf\{V_t(z), z\in G(y_0)\}+2\varepsilon.$

Contents

A Generalization

We put $Z = G(y_0)$, and $\bar{Z}$ its closure

Theorem 8 ***(H1)*** h ***is uniformly continuous in*** y ***on*** $\bar{Z}$ ***uniformly in*** u***. And for each*** y ***in*** $\bar{Z}$***, either*** h ***does not depend on*** u ***or the set*** $\{(g(y,u), h(y,u)) \in IR^d \times [0,1],\ u \in U\}$ ***is closed.***
(H2): $\exists \Delta : IR^d \times IR^d \longrightarrow IR_+$***, vanishing on the diagonal*** ***(***$\Delta(y,y) = 0$***) and symmetric (***$\Delta(y_1,y_2) = \Delta(y_2,y_1)$ ***), and a function*** $\hat{\alpha} : IR_+ \longrightarrow IR_+$ ***s.t.*** $\hat{\alpha}(t) \underset{t \to 0}{\longrightarrow} 0$ ***satisfying:***

$a)$ $\forall$ ***sequence*** $(z_n)_n \subset Z$***,*** $\forall \varepsilon > 0$***,*** $\exists n$***,*** $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$***.***

$b)$ $\forall (y_1, y_2) \in \bar{Z}^2$, $\forall u \in U$***,*** $\exists v \in U$ ***such that***

$$D \uparrow \Delta(y_1,y_2)(g(y_1,u), g(y_2,v)) \leq 0, \ h(y_2,v) - h(y_1,u) \leq \hat{\alpha}(\Delta(y_1,y_2)).$$

Then $\Gamma(y_0)$ ***has a uniform value*** $\lim_{t \to \infty} V_t = V^*$.

Remarks

• **First result corresponds to the case where:** $\Delta(y_1, y_2) = \|y_1 - y_2\|^2$**,** $G(y_0)$ **is bounded, and** $h(y, u) = h(y)$ **(one can just take** $\hat{\alpha}(t) = \sup\{|h(x) - h(y)|, \|x - y\|^2 \leq t\}$**).**

• • **Although** Δ **may not satisfy the triangular inequality nor the separation property, it may be seen as a "distance" adapted to the problem** $\Gamma(y_0)$**.**

• $D \uparrow$ **is the contingent epi-derivative (which reduces to the upper Dini derivative if** Δ **is Lipschitz)** $D{\uparrow}\Delta(z)(\alpha) = \liminf_{t\to 0^+, \alpha'\to\alpha} \frac{1}{t}(\Delta(z + t\alpha') - \Delta(z))$**. If** Δ **is differentiable, the condition** $D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ **just reads:** $< g(y_1, u), \frac{\partial}{\partial y_1}\Delta(y_1, y_2) > + < g(y_2, v), \frac{\partial}{\partial y_2}\Delta(y_1, y_2) > \leq 0$**.**

• **The assumption: "**$\{(g(y,u), h(y,u)) \in I\!R^d \times [0,1],\ \ u \in U\}$ **closed" could be checked for instance if** U **is compact and if** h **and** g **are continuous with respect to** (y,u)**.**

• $H2a)$ **is a precompacity condition. It is satisfied as soon as** $G(y_0)$ **is bounded. cf Renault 2008**

• **Notice that** $H2$ **is satisfied with** $\Delta = 0$ **if we are in the trivial case where** $\inf_u h(y,u)$ **is constant.**

• **Theorem 8 can be applied to example 4, with** $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ **(**L^1**-norm). In this example, we have for each** y_1**,** y_2 **and** u**:** $\Delta(y_1 + tg(y_1,u), y_2 + tg(y_2,u)) \leq \Delta(y_1, y_2)$ **as soon as** $t \geq 0$ **is small enough.**

Proposition 9 ***We have***

$$\begin{cases} \forall (y_1, y_2) \in \bar{Z}^2,\ \forall T \geq 0,\ \forall \varepsilon > 0, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U}, \\ \forall t \in [0,T],\ \Delta(y(t,u,y_1), y(t,v,y_2)) \leq \Delta(y_1,y_2) + \varepsilon, \\ \quad \textbf{\textit{and for almost every }} t \in [0,T]\textbf{\textit{,}} \\ h(y(t,v,y_2), v(t)) - h(y(t,u,y_1), u(t)) \leq \hat{\alpha}(\Delta(y(t,u,y_1), y(t,v,y_2))). \end{cases}$$

Corollary 10 ***For every*** y_1 ***and*** y_2 ***in*** $G(y_0)$***, for all*** $T > 0$***,***

$$|V_T(y_1) - V_T(y_2)| \leq \hat{\alpha}(\Delta(y_1, y_2)).$$

Lemma 11 ***For every*** $\varepsilon > 0$***, there exists*** m_0 ***in*** $I\!R_+$ ***such that:***

$$\forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \ \ \textbf{\textit{such that}} \ \ \Delta(z,z') \leq \varepsilon.$$

Proposition 12 $V_t(y_0) \xrightarrow[t\to\infty]{} V^*(y_0).$

Definition 13 $\Gamma(y_0)$ ***has a*** *uniform value* ***if*** $\exists V(y_0)$ ***and if:***

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds \leq V(y_0)+\varepsilon.$$

Given $m \geq 0$ **and** $n \geq 1$**,** $\forall z \in Z = G(y_0)$ $\forall u \in \mathcal{U}$**, we define**

$$\nu_{m,n}(z,u) = \sup_{t\in[1,n]} \gamma_{m,t}(z,u), \text{ and } W_{m,n}(z) = \inf_{u\in\mathcal{U}} \nu_{m,n}(z,u).$$

$W_{m,n}$ **is the value function of the problem where the controller can use the time interval** $[0,m]$ **to reach a "good state", and then his cost is only the supremum for** t **in** $[1,n]$**, of the average cost between time** m **and** $m+t$**.**

Obviously $W_{m,n} \geq V_{m,n}$**.**

Uniform convergence and existence of a Uniform Value

- **STEP 1 For every z and z' in Z, for all $m \geq 0$ and $n \geq 1$,** $|V_{m,n}(z) - V_{m,n}(z')| \leq \hat{\alpha}(\Delta(z, z')), |W_{m,n}(z) - W_{m,n}(z')| \leq \hat{\alpha}(\Delta(z, z'))$.
- **STEP 2** $\forall k, n \geq 1, \forall m, \forall z \in Z,\ V_{m,n}(z) \geq \inf_{l \geq m} W_{l,k}(z) - \frac{k}{n}$.
- **STEP 3** $\forall z \in Z,\ \inf_{m \geq 0} \sup_{n \geq 1} W_{m,n}(z) = \inf_{m \geq 0} \sup_{n \geq 1} V_{m,n}(z)$
$$= V^*(z) = \sup_{n \geq 1} \inf_{m \geq 0} V_{m,n}(z) = \sup_{n \geq 1} \inf_{m \geq 0} W_{m,n}(z).$$
- **STEP 4 The convergence of $(V_n)_n$ to V^* is uniform on Z.**
- **STEP 5** $\forall \varepsilon > 0, \exists M \geq 0, \exists K \geq 1, \forall z \in Z, \exists m \leq M, \forall n \geq K, \exists u \in \mathcal{U}$ **such that:**
$$\nu_{m,n}(z, u) \leq V^*(z) + \varepsilon/2, \text{ and } V^*(y(m+n, u, z)) \leq V^*(z) + \varepsilon.$$

Optimal control with discounted facteur $\lambda \to 0^+$

We define $\Theta_\lambda(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s,u,y_0), u(s))ds,$

Recall the Technical Lemma :$\forall m_0 \geq 0$**, we have:**

$$\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq V^+(y_0) \geq V^-(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

Define $\Theta^-(y_0) := \liminf_{\lambda \to 0} \Theta_\lambda(y_0), \Theta^+(y_0) := \limsup_{\lambda \to 0} \Theta_\lambda(y_0).$

Lemma 14 $\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq \Theta^+(y_0) \geq \Theta^-(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$

Theorem 15 ***Under the assumptions of Theorem 5 or 8, the limit*** $\lim_{\lambda \to 0^+} \Theta_\lambda(y_0)$ ***exists.***

Contents

An Open Problem

Differential Game at horizon t:

$$V_t(y_0) := \text{"} \inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} \text{"} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,v,y_0),u(s),v(s))ds,$$

where $s \mapsto y(s,u,y_0)$ denotes the solution to

$$y'(s) = g(y(s),u(s),v(s))), \quad y(0) = y_0.$$

OPEN PROBLEM : Existence of a limit of $V_t(y_0)$ as $t \to \infty$.

Only Partial results:

- **When the Hamiltonian is coercive (hence ergodicity and the limit is y independent) Alvarez-Bardi ...**
- **For nonconvex and non coercive Hamiltonian in $I\!R^2$ Cardaliagu**

References

[1] O. Alvarez, M. Bardi *Ergodicity, Stabilization, and singular perturbations for Bellman-Isaacs equations* To appear in Mem AMS.

[2] Z. Artstein, and V. Gaitsgory, *The value function of singularly perturbed control systems*, Appl. Math. Optim., 41 (2000), 425-445.

[3] Arisawa, M. and P.L. Lions (1998) Ergodic problem for the Hamilton Jacobi Belmann equations II, *Ann. Inst. Henri Poincaré, Analyse Nonlinéaire*, 15 ,1 , 1–24.

[4] Arisawa, M. and P.L. Lions (1998) On ergodic stochastic control. *Com. in partial differential equations*, 23, 2187–2217.

[5] Bettiol, P. (2005) On ergodic problem for Hamilton-Jacobi-Isaacs equations *ESAIM: Cocv*, 11, 522–541.

[6] Cardaliaguet P. Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamiltonian in $I\!R^2/Z^2$ preprint [hal-00348219 - version 1] (18/12/2008)

[7] Frankowska, H., Plaskacz, S. and Rzezuchowski T. (1995): Measurable Viability Theorems and Hamilton-Jacobi-Bellman Equation, J. Diff. Eqs., 116, 265-305.

[8] V. Gaitsgory, *Use of Averaging Method in Control Problems*, Differential Equations (Translated from Russian), 46 (1986) pp. 1081-1088.

[9] V. Gaitsgory, M. Quincampoix *Linear programming analysis of deterministic infinite horizon optimal control problems (discounting and time averaging cases)*, to appear in Siam Journal of Control and Opt.

[10] G. Grammel, *Averaging of singularly perturbed systems*, Nonlinear Analysis Theory, 28, 11 (1997), 1851-1865.

[11] Lions P.-L. , Papanicolaou G. , Varadhan S.R.S., Homogenization of Hamilton-Jacobi Equations, unpublished work.

[12] Quincampoix, M. and F. Watbled (2003) Averaging methods for discontinuous Mayer's problem of singularly perturbed control systems. *Nonlinear analysis*, 54, 819–837.

[13] Quincampoix, M. and J. Renault (submitted) On Existence of a limit value in some non expansive optimal control problems, and application to averaging of singularly perturbed systems.

[14] Renault, J. (2007) Uniform value in Dynamic Programming. Cahier du Ceremade 2007-1. arXiv : 0803.2758.

Thank You for your Attention