Asymptotic behavior of values of zero-sum repeated games: evolution equations in discrete and continuous time

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Table of contents

Table of contents

Definition

A zero-sum stochastic game is a 5-tuple (Ω, A, B, g, ρ) where:

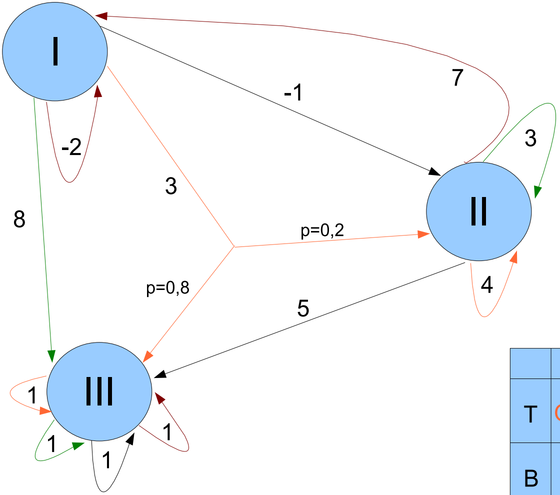
- Ω is the set of states
- A (resp. B) is the action state of J_1 (resp. J_2)
- $g : A \times B \times \Omega \to \mathbb{R}$ is the payoff function, which will be assumed bounded.
- $\rho : A \times B \times \Omega \to \Delta(\Omega)$ is the transition probability.

How the Game is played

An initial state ω_1 is given, known by each player.
During each stage i:

- the players observe the current state ω_i.
- According to the past history, J_1 (resp. J_2) choose a mixed action x_i in $\Delta(A)$ (resp. y_i in $\Delta(B)$).
- An action a_i of player 1 (resp. b_i of player 2) is drawn according to his mixed strategy x_i (resp. y_i).
- This gives the payoff at stage i $g_i = g(a_i, b_i, \omega_i)$.
- A new state ω_{i+1} is drawn according to $\rho(a_i, b_i, \omega_i)$.

Example



	L	R
T	Orange	Green
B	Red	Black

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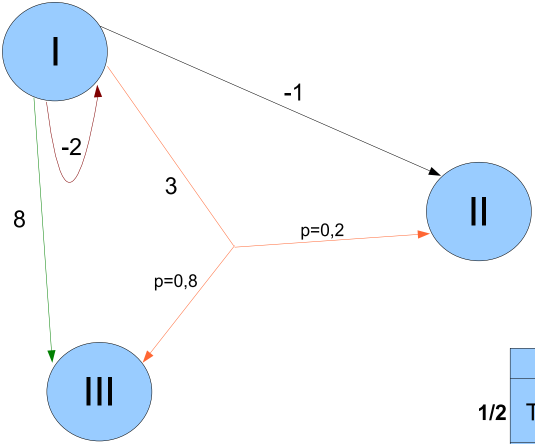

		1/4	3/4
		L	R
1/2	T	1/8	3/8
1/2	B	1/8	3/8

Payoff of the repeated game

There are several ways of evaluating a payoff for a given infinite history :

- $\frac{1}{n}\sum_{i=1}^{n} g_i$ is the payoff of the $n-$stage game
- $\lambda\sum_{i=1}^{+\infty}(1-\lambda)^{i-1}g_i$ is the payoff of the $\lambda-$discounted game.

If those games have a value for a given initial state ω, we denote them by $v_n(\omega)$ and $v_\lambda(\omega)$ respectively.
Thus v_n and v_λ are functions from Ω into $\mathbb{R}$.

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Asymptotic behavior

The main problem which arises is the study of the behavior of v_n when $n \to +\infty$ and of v_λ when $\lambda \to 0$. Does the limits exist, and are they the same ?

We know that the answers to both questions are positive in several cases :

- (Bewley-Kohlberg) Finite stochastic games (Ω, A and B finite)
- (Kohlberg) Absorbing games
- (Everett) Recursive games
- (Aumann-Maschler, Mertens-Zamir) Games with incomplete information and standard signaling
- (Renault) Markov Chain Games with lack of information on one side.

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Table of contents

Recursive structure

The values of the finitely repeated game and of the discounted game satisfy a recursive structure:

$$
\begin{aligned}
v_n(\omega) &= \sup_{x\in\Delta(A)} \inf_{y\in\Delta(B)} \left\{ \frac{1}{n} g(x,y,\omega) + \frac{n-1}{n} E_{\rho(x,y,\omega)}(v_{n-1}) \right\} \\
&= \inf_{y\in\Delta(B)} \sup_{x\in\Delta(A)} \left\{ \frac{1}{n} g(x,y,\omega) + \frac{n-1}{n} E_{\rho(x,y,\omega)}(v_{n-1}) \right\} \\
v_\lambda(\omega) &= \sup_{x\in\Delta(A)} \inf_{y\in\Delta(B)} \left\{ \lambda g(x,y,\omega) + (1-\lambda) E_{\rho(x,y,\omega)}(v_\lambda) \right\} \\
&= \inf_{y\in\Delta(B)} \sup_{x\in\Delta(A)} \left\{ \lambda g(x,y,\omega) + (1-\lambda) E_{\rho(x,y,\omega)}(v_\lambda) \right\}
\end{aligned}
$$

Shapley operator

Let $\mathscr{F}$ be the set of bounded functions from Ω into $\mathbb{R}$ and define $\Psi : \mathscr{F}$ to itself by

$$\begin{aligned}\Psi(f)(\omega) &= \sup_{x\in\Delta(A)} \inf_{y\in\Delta(B)} \left\{g(x,y,\omega)+E_{\rho(x,y,\omega)}(f)\right\} \\ &= \inf_{y\in\Delta(B)} \sup_{x\in\Delta(A)} \left\{g(x,y,\omega)+E_{\rho(x,y,\omega)}(f)\right\}.\end{aligned}$$

Consider also the family of operators $\Phi(\alpha,\cdot)$ for $\alpha \in]0,1]$ defined by the formula $\Phi(\alpha,f) = \alpha\Psi\left(\frac{1-\alpha}{\alpha}f\right)$.
Then the recursive equations can be written as

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right) = \frac{\Psi^n(0)}{n}$$

$$v_\lambda = \Phi(\lambda, v_\lambda)$$

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Properties of Ψ and $\Phi(\alpha,\cdot)$

The operator Ψ is topical : it satisfies the two following properties :

- Monotonicity $\qquad f_1 \leq f_2 \implies \Psi(f_1) \leq \Psi(f_2)$
- Homogeneity $\qquad c \in \mathbb{R} \implies \Psi(f+c) = \Psi(f) + c$

These two properties implies that Ψ is nonexpansive for the infinite norm

$$\|\Psi(f) - \Psi(g)\|_\infty \leq \|f - g\|_\infty$$

and thus that $\Phi(\alpha,\cdot)$ is $1-\alpha$ contracting for the infinite norm

$$\|\Phi(\alpha,f) - \Phi(\alpha,g)\|_\infty \leq (1-\alpha)\|f - g\|_\infty$$

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Operator approach (Sorin Rosenberg '01)

Let $(X, \|\|)$ be a Banach space, and let $\Psi : X \to X$ be an nonexpansive operator.

Let us define the family of contracting operators $\Phi(\alpha, \cdot) : X \to X$ by the formula

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Questions

We now settle the following questions :

- Does $\lim_{n \to +\infty} v_n$ exist ?
- Does $\lim_{\lambda \to 0} v_\lambda$ exist ?
- Are those two limits equal ?
- Identification of the limit ?

Table of contents

When λ is fixed

Recall that for any u_0 in X, $v_\lambda = \Phi^\infty(\lambda, u_0)$

Proposition

When λ is fixed, the solution u of the evolution equation

$$u(t) + u'(t) = \Phi(\lambda, u(t)) \quad ; \quad u(0) = u_0 \in X \tag{1}$$

satisfies

$$\lim_{t \to +\infty} u(t) = v_\lambda$$

Sketch of proof

Lemma

The solution of (1) satisfies $\|u(t)-v_\lambda\| \leq \frac{\|u'(t)\|}{\lambda}$.

Lemma

If f satisfies $\|f(t)+f'(t)\| \leq (1-\lambda(t))\|f(t)\|$, then

$$\|f(T)\| \leq \|f(0)\| e^{-\int_0^T \lambda(t)dt}.$$

Let us apply the second lemma to $f_h = \frac{u(t+h)-u(t)}{h}$, so that $\|f_h(t)\| \leq \|f_h(0)\| e^{-\lambda t}$. We then let h go to 0 and we use the first lemma.

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Non autonomous case

We are now interested in the equation of the type

$$u(t)+u'(t)=\Phi(\lambda(t),u(t)) \quad ; \quad u(0)=u_0\in X \tag{2}$$

where λ is a continuous function from $\mathbb{R}^+$ into $]0,1[$.
If the parametrization λ converge slowly enough to 0, because of the previous result we expect the trajectory to be asymptotically close to the location of the fixed points $v\lambda$.

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Any accumulation point v of v_n or v_λ is a fixed point of $\Phi(0,\cdot)$, but there may be many of these (for example in games with incomplete information on both sides any saddle function). The evolution equation (2) can be viewed as a perturbation of

$$u(t)+u'(t)=\Phi(0,u(t))$$

and if the perturbation is strong enough it will select a "good" fixed point (see also Attouch Cominetti '96 ; Cominetti Peypouquet Sorin '08)

Proposition

If $\lambda \notin \mathscr{L}^1$, then the asymptotic behavior of the solution of (2) doesn't depend of u_0.

Hypothesis on $\Phi(\cdot,x)$

From now on we assume that:

Hypothesis

$\exists C \in \mathbb{R},\ \forall(\lambda,\mu) \in]0,1[^2,\ \forall x \in X,$

$$\|\Phi(\lambda,x) - \Phi(\mu,x)\| \leq C|\lambda - \mu|(1 + \|x\|) \qquad (\mathscr{H})$$

Remark

This hypothesis is satisfied as soon as Ψ is the Shapley operator of any game with a bounded payoff.

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Consequences (I)

Proposition

Let λ and $\widetilde{\lambda}$ be two parametrization, and let u and $\widetilde{u}$ be the corresponding solutions of (2). If $\lambda \notin \mathscr{L}^1$, if u is bounded and if $\lambda(t) \sim \widetilde{\lambda}(t)$ then $\lim_{t\to+\infty} \|u(t) - \widetilde{u}(t)\| = 0$

Corollary

If $\lambda(t) \to \lambda_0 > 0$ then $u(t) \to v_{\lambda_0}$.

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Consequences(II)

Proposition

If $\lambda \downarrow 0$ *is in* $\mathscr{C}^1$ *and if* $\lim_{t\to+\infty} \frac{\lambda'(t)}{\lambda^2(t)} = 0$, *then* $\|u(t) - v_{\lambda(t)}\| \to 0$

If $\lim_{t\to+\infty} \frac{\lambda''(t)}{\lambda(t)\lambda'(t)} = 0$ *then the rate of convergence is in* $O\left(\frac{\lambda'(t)}{\lambda^2(t)}\right)$.

Corollary

If $\lambda(t) \sim \frac{1}{t^\alpha}$ *for an* $\alpha \in]0,1[$ *then* $\left\|u(t) - v_{\lambda(t)}\right\| \to 0$.
In particular v_λ *converges when* $\lambda \to 0$ *if and only if* $u(t)$ *converges when* $t \to +\infty$.

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Back to discrete time

For every λ_n sequence of numbers in $]0,1[$ let us define the sequence w_n of element of X by

$$w_n = \Phi(\lambda_n, w_{n-1})$$

Proposition

If $\lambda_n \to 0$ and $\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \to 0$, then $\|w_n - v_{\lambda_n}\| \to 0$

Corollary

If $\lambda_n \to 0$, $\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \to 0$, and if w_n converges, then v_λ converges to the same limit.

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Table of contents

Evolution equation related to V_n

Let us denote $V_n = nv_n = \Psi^n(0) = \Psi(V_{n-1})$.

We consider the differential equation

$$U(t) + U'(t) = \Psi(U(t)) \quad ; \quad U(0) = 0. \tag{3}$$

that is

$$U'(t) = -A(U(t)) \quad ; \quad U(0) = 0.$$

where we denote by A the maximal monotone operator $Id - \Psi$.

Proposition (Miyadera Oharu '70)

The solution of (3) satisfies

$$\|U(n) - V_n\| \leq \sqrt{n} \cdot \|\Psi(0)\|.$$

$$\left\| \frac{U(n)}{n} - v_n \right\| \to 0.$$

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The solution of (3) satisfies

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Proposition

Let $\tau(t) = t + \ln(1+t)$, and let u be the solution of the evolution equation

$$u(t) + u'(t) = \Phi\left(\frac{1}{2+\tau^{-1}(t)}, u(t)\right).$$

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