THEOREMS OF THE ALTERNATIVE FOR MULTIVALUED MAPPINGS.

APPLICATIONS TO MIXED CONVEX\CONCAVE SYSTEMS OF INEQUALITIES

Michel Volle

University of Avignon

X (real) linear space $f_1, \dots, f_n \in X^*$

GORDAN ALTERNATIVE (1873)

$\exists x \in X : f_i(x) < 0 \, , \forall i \in \{1, \dots, n\}$

$\exists (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n \setminus \{0\} \; : \; \sum_{i=1}^{n} \alpha_i f_i(x) = 0 \, , \forall x \in X$

FARKAS ALTERNATIVE (1902)

$\exists x \in X : f_1(x) < 0 \, , f_2(x) \leq 0, \dots, f_n(x) \leq 0$

$\exists \alpha_1 > 0, \alpha_2 \geq 0, \dots, \alpha_n \geq 0 : \sum_{i=1}^{n} \alpha_i f_i(x) = 0 \, , \forall x \in X$

FAN-GLICKSBER-HOFFMAN ALTERNATIVE (1957)

$X \supset C$ convex $\quad f_1, \dots, f_n : X \to \mathbb{R}$ convex

$\exists x \in C : f_i(x) < 0 \, , \forall i \in \{1, \dots, n\}$

$\exists (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n \setminus \{0\} \; : \; \sum_{i=1}^{n} \alpha_i f_i(x) \geq 0 \, , \forall x \in C$

SCHIROTZEK ALTERNATIVE (2007)

X, Z_1, Z_2 normed spaces, $X \supset C$ convex $Z_i \supset S_i$ convex cones $(i=1,2)$

$$H_i : C \to Z_i \quad S_i\text{-convex} \quad (i=1,2)$$

$$\operatorname{int} S_1 \neq \emptyset \quad S_2 \text{ closed}$$

$$(1) \qquad \exists a \in Z_1 : (a, 0) \in \operatorname{int}\Big(\bigcup_{x \in C} \{H_1(x)\} \times \{H_2(x)\} + S_1 \times S_2 \Big)$$

$$\nexists x \in C : H_1(x) \in \operatorname{int}(-S_1),\ H_2(x) \in -S_2$$

$$\exists \lambda_1 \in S_1^+ \setminus \{0\},\ \exists \lambda_2 \in S_2^+ : (\lambda_1 \circ H_1)(x) + (\lambda_2 \circ H_2)(x) \geq 0,\ \forall x \in C$$

<u>Remark</u>: (1) is satisfied if (2) holds:

$$(2) \quad \exists \bar{x} \in C : H_2(\bar{x}) \in \operatorname{int}(-S_2)$$

MULTIVALUED APPROACH

X linear space, Z_1, Z_2 t.v.s., $X \supset C$ a set, $Z_i \supset S_i$ two convex cones, $\mathrm{int}\, S_1 \neq \phi$,

$$M_i : \mathrm{dom}\, M_i \subset X \rightrightarrows Z_i \quad \text{two multivalued maps} \quad (i=1,2)$$

$$D = C \cap \mathrm{dom}\, M_1 \cap \mathrm{dom}\, M_2$$

(σ) $\quad \exists x \in D : M_1(x) \cap \mathrm{int}(-S_1) \neq \phi \,, \; M_2(x) \cap (-S_2) \neq \phi$

BACK TO UNIVOQUE CASE

Four possibilities: $M_i(x) = \{H_i(x)\}$ or $M_i(x) = H_i(x) + S_i \quad (i=1,2)$

LOOKING FOR AN ALTERNATIVE FORMULATION FOR (σ)

Notations: the nonnegative polar cone S_i^+ of S_i $(i=1,2)$

$$S_i^+ = \{ d_i \in Z_i' : \langle d_i, z_i \rangle = d_i(z_i) \geq 0, \forall z_i \in S_i \}$$

$$(\sigma) \qquad \exists x \in D : M_1(x) \cap \operatorname{int}(-S_1) \neq \phi \,, \quad M_2(x) \cap (-S_2) \neq \phi$$

$$(\sigma') \quad \exists \lambda_1 \in S_1^+ \setminus \{0\}, \exists \lambda_2 \in S_2^+ : \langle \lambda_1, z_1 \rangle + \langle \lambda_2, z_2 \rangle \geqslant 0, \ \forall x \in D, \forall z_i \in M_i(x), \ i = 1,2$$

Recall $\quad X \supset C \quad M_i : \operatorname{dom} M_i \subset X \rightrightarrows Z_i \quad Z_i \supset S_i$ convex cones $\quad D = C \cap \operatorname{dom} M_1 \cap \operatorname{dom} M_2$

Without convexity nor topological requirement on C, M_1, M_2, one has

<u>Pr 1</u> If (σ') holds then (σ) is unconsistent

To go further one needs convexity <u>somewhere</u> and a so-called topological qualification condition

Convexity somewhere

$$E := S_1 \times S_2 + \bigcup_{x \in D} M_1(x) \times M_2(x) \text{ must be convex } \quad (D = C \cap \operatorname{dom} M_1 \cap \operatorname{dom} M_2)$$

Rq1 $\quad E$ may be convex while $\bigcup_{x \in D} M_1(x) \times M_2(x)$ is not

Ex1: $M_i(x) = \{H_i(x)\}$, H_i S_i-convex $(i = 1,2)$

Standard situation : C convex, M_1, M_2 convex $\Rightarrow \bigcup_{x \in D} M_1(x) \times M_2(x)$ convex

$\Downarrow$

Ex2 : $M_i(x) = H_i(x) + S_i$, H_i S_i-convex $(i=1,2)$ $\qquad$ E convex

Nonstandard situation : $\bigcup_{x \in D} M_1(x) \times M_2(x)$ may be convex while C, M_1, M_2 are not!

Ex3 (Brickman, 1961) $\quad X = \mathbb{R}^n$, C euclidean unit sphere, $M_i(x) = \{\langle A_i x, x\rangle\}$,

A_1, A_2 real symmetrics $n \times n$ matrices

$$\bigcup_{x \in C} \{\langle A_1 x, x\rangle\} \times \{\langle A_2 x, x\rangle\} \text{ convex compact in } \mathbb{R}^2$$

Ex4 (<u>Dines</u>, 1941) $M_i(x) = \{\langle A_i x, x\rangle\}$ $A_1, A_2 \in \mathcal{S}_n(\mathbb{R})$

$$\bigcup_{x \in \mathbb{R}^n} \{\langle A_1 x, x\rangle\} \times \{\langle A_2 x, x\rangle\} \text{ convex cone in } \mathbb{R}^2$$

<u>Topological requirement</u> (remember: $E = S_1 \times S_2 + \bigcup_{x \in D} M_1(x) \times M_2(x)$)

$$\exists a \in Z_1 : (a, 0) \in \operatorname{int} E$$

which holds in particular if (remember: $D = C \cap \operatorname{dom} M_1 \cap \operatorname{dom} M_2$)

$$\exists \bar{x} \in D : M_2(\bar{x}) \cap \operatorname{int}(-S_2) \neq \emptyset$$

<u>TH1</u> Assume $E = S_1 \times S_2 + \bigcup_{x \in D} M_1(x) \times M_2(x)$ is convex and

$\exists a \in Z_1 : (a, 0) \in \operatorname{int} E$

Then precisely one of the following statements is true

(σ) $\exists x \in D : M_1(x) \cap \operatorname{int}(-S_1) \neq \phi,\ M_2(x) \cap \operatorname{int}(-S_2) \neq \phi$

(σ') $\exists \lambda_1 \in S_1^+ \setminus \{0\}, \exists \lambda_2 \in S_2^+ : \langle \lambda_1, z_{1} \rangle + \langle \lambda_2, z_{2} \rangle \geqslant 0,\ \forall x \in D,\ \forall z_{i} \in M_i(x)\ (i = 1, 2)$

<u>COR1</u> Assume $E = S_1 \times S_2 + \bigcup_{x \in D} \{H_1(x)\} \times \{H_2(x)\}$ is convex and

$\exists a \in Z_1 : (a, 0) \in \operatorname{int} E$

Then precisely one of the following statements is true

$\exists x \in D : H_1(x) \in \operatorname{int}(-S_1),\ H_2(x) \in -S_2$

$\exists \lambda_1 \in S_1^+ \setminus \{0\}, \exists \lambda_2 \in S_2^+ : (\lambda_1 \circ H_1)(x) + (\lambda_2 \circ H_2)(x) \geqslant 0,\ \forall x \in D$

Example: Systems of mixed large and strict convex inequalities

X linear space, $X \supset C$ convex

$$f_1, \dots, f_p, g_1, \dots, g_q : X \to \mathbb{R} \cup \{+\infty\} \text{ convex}$$

$$H_1(x) = (f_1(x), \dots, f_p(x)), \quad \operatorname{dom} H_1 = \bigcap_{1 \le i \le p} \operatorname{dom} f_i = F$$

$$H_2(x) = (g_1(x), \dots, g_q(x)) \quad \operatorname{dom} H_2 = \bigcap_{1 \le j \le q} \operatorname{dom} g_j = G$$

Assume that $\exists \bar{x} \in C \cap F \cap G : g_j(\bar{x}) < 0, \ \forall j = 1, \dots, p$

Then precisely one of the following statements is true

$$\exists x \in C : f_1(x) < 0, \dots, f_p(x) < 0, \ g_1(x) \le 0, \dots, g_q(x) \le 0$$

$$\exists \alpha_1 \ge 0, \dots, \alpha_p \ge 0, \text{ non all } 0, \ \exists \beta_1 \ge 0, \dots, \exists \beta_q \ge 0 : \sum_{i=1}^{p} \alpha_i f_i(x) + \sum_{j=1}^{q} \beta_j g_j(x) \ge 0, \ \forall x \in C \cap F \cap G$$

The case when only M_1 occurs : take $S_2 = Z_2$, $M_2(x) = 0$, $\forall x \in X$

<u>COR 2</u> Let X a linear space, C a subset of X, S_1 a convex cone in the t.v.s. Z_1, $\operatorname{int} S_1 \neq \emptyset$, and $M_1 : \operatorname{dom} M_1 \subset X \rightrightarrows$ a multivalued mapping such that

$$M_1(C \cap \operatorname{dom} M_1) + S_1 \text{ is convex}$$

Then precisely one of the following statements is true

(σ) $\exists x \in C \cap \operatorname{dom} M_1 : M_1(x) \cap \operatorname{int}(-S_1) \neq \emptyset$

(σ') $\exists \lambda_1 \in S_1^+ \setminus \{0\} : \langle \lambda_1, z_1 \rangle \geq 0, \forall x \in C \cap \operatorname{dom} M_1, \forall z_1 \in M_1(x)$

Univoque case $H_1 : \operatorname{dom} H_1 \subset X \to Z_1$ $H_1(C \cap \operatorname{dom} H_1) + S_1$ convex

(σ) $\exists x \in C \cap \operatorname{dom} H_1 : H_1(x) \in \operatorname{int}(-S_1)$

(σ') $\exists \lambda_1 \in S_1^+ \setminus \{0\} : (\lambda_1 \circ H_1)(x) \geq 0, \forall x \in C \cap \operatorname{dom} H_1$

Ex 1 $\quad Z_1 = \mathbb{R}^n \quad S_1 = \mathbb{R}^n_+ \qquad f_1, \dots, f_n : X \to \mathbb{R} \cup \{+\infty\}$

$H_1(x) = (f_1(x), \dots, f_n(x)), \ \forall x \in \bigcap_{1 \le i \le n} \operatorname{dom} f_i = \operatorname{dom} H_1 = F$

Assume C is convex in X and $f_1, \dots, f_n$ are convex. Then

$$\left.\begin{array}{l} \exists x \in C : \ f_i(x) < 0, \ \forall i \in \{1, \dots, n\} \\ \exists (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{0\} : \sum_{i=1}^{n} \alpha_i f_i(x) \geq 0, \ \forall x \in C \cap F \end{array}\right\} \ \frac{\text{F\G\H alternative}}{(1957)}$$

Ex 2 $\quad C = X = \mathbb{R}^n, \ Z_1 = \mathbb{R}^2, \ S_1 = \mathbb{R}^2_+, \ H_1(x) = (\langle Ax, x\rangle, \langle Bx, x\rangle), \ A, B \in \mathcal{S}_n(\mathbb{R})$

$H_1(C)$ is a convex cone in $\mathbb{R}^2$ (<u>Dines THM</u>) $\Rightarrow$ $H_1(C) + S_1$ convex (cone). Thus

$$\text{non}(\sigma) \iff (\sigma')$$

$$\max\{\langle Ax, x\rangle, \langle Bx, x\rangle\} \geq 0, \ \forall x \in \mathbb{R}^n \iff \exists \alpha \geq 0, \exists \beta \geq 0 : \alpha + \beta = 1 \text{ and } \alpha A + \beta B \text{ positive semidefinite}$$

This is <u>Yuan alternative</u> (1990)

A variant of Yuan alternative

Back to Cor. 1 $\quad D = X = \mathbb{R}^n \quad Z_1 = Z_2 = \mathbb{R} \quad S_1 = S_2 = \mathbb{R}_+$

$H_1(x) = \langle Ax, x\rangle \quad H_2(x) = \langle Bx, x\rangle \quad A, B \in \mathcal{S}_n(\mathbb{R})$

Fact : $\bigsqcup_{x \in D} \{H_1(x)\} \times \{H_2(x)\}$ is a convex cone (Dines Thm)

Assume $\exists\, \bar{x} \in \mathbb{R}^n : \langle B\bar{x}, \bar{x}\rangle < 0$ (Topological condition)

Then

$\exists x \in \mathbb{R}^n : \langle Ax, x\rangle < 0$ and $\langle Bx, x\rangle \leq 0$

$\exists \lambda_1 > 0, \exists \lambda_2 \geq 0 : \lambda_1 A + \lambda_2 B$ s.d.p. (or $\exists \beta \geq 0 : A + \beta B$ s.d.p.)

Compare with Yuan alternative : $\forall A, B \in \mathcal{S}_n(\mathbb{R})$

$\exists x \in \mathbb{R}^n : \langle Ax, x\rangle < 0$ and $\langle Bx, x\rangle < 0$

$\exists \alpha \geq 0, \exists \beta \geq 0 : \alpha + \beta = 1$ and $\alpha A + \beta B$ s.d.p.

ALTERNATIVE INVOLVING MULTIVALUED MAPPINGS, HIT, AND CONTAINMENT

Datas X, Z_1, Z_2 t.v.s, $C \subset X$, $M_i : \mathrm{dom}\, M_i \subset X \rightrightarrows Z_i$

S_i convex cone in Z_i, $\mathrm{int}\, S_1 \neq \phi$, $D = C \cap \mathrm{dom}\, M_1 \cap \mathrm{dom}\, M_2$

(γ) $\quad \exists x \in D : M_1(x) \cap \mathrm{int}(-S_1) \neq \phi \,, \quad M_2(x) \subset S_2$

Univoque case: $M_i(x) = \{H_i(x)\}$

(γ_1) $\quad \exists x \in D : H_1(x) \in \mathrm{int}(-S_1) \,, \quad H_2(x) \in S_2$

Ex $Z_1 = \mathbb{R}^p$ $S_1 = \mathbb{R}^p_+$ $Z_2 = \mathbb{R}^q$ $S_2 = \mathbb{R}^q_+$ $H_1(x) = (f_1(x), \dots, f_p(x))$

$H_2(x) = (g_1(x), \dots, g_q(x))$

(γ_2) $\quad \exists x \in D : f_1(x) < 0, \dots, f_p(x) < 0, \; g_1(x) \geqslant 0, \dots, g_q(x) \geqslant 0$

System of mixed convex\concave inequalities

USING THE S_2-SUBDIFFERENTIAL OF $M_2 : \mathrm{dom}\, M_2 \subset X \rightrightarrows Z_2$

Def Given $\bar{x} \in \mathrm{dom}\, M_2$ one says that $L \in \mathcal{L}(X, Z_2)$ is a S_2-subgradient of M_2 at $\bar{x}$ (we note $L \in \partial M_2(\bar{x})$) if there exists $\bar{z} \in M_2(\bar{x})$ s.t.

$$z - \bar{z} - L(x - \bar{x}) \in S_2 \,, \quad \forall (x, z) \in \mathcal{G}(M_2)$$

Univoque case $M_2(x) = \{H_2(x)\}$

$$H_2(x) - H_2(\bar{x}) - L(x - \bar{x}) \in S_2 \,, \quad \forall x \in \mathrm{dom}\, H_2$$

$L \in \partial H_2(\bar{x})$ in the "usual" sense

Other example $M_2(x) = H_2(x) + S_2$

again $\partial M_2(\bar{x}) = \partial H_2(\bar{x})$

Many other examples may be provided

USING THE <u>EXACT CONJUGATE</u> OF $M_2 : \operatorname{dom} M_2 \subset X \rightrightarrows Z_2$

Assume that the convex cone $S_2 \subset Z_2$ satisfies

$$S_2 \cap (-S_2) = \{0\}$$

Then M_2 is S_2-subdifferentiable at $\bar{x}$ iff

$M_2(\bar{x})$ admits a least element and $\exists L \in \mathcal{L}(X, Z_2) : z - \min M_2(\bar{x}) \geq L(x - \bar{x}), \forall (x, z) \in \mathcal{G}(M_2)$

For any $L \in \bigcup_{x \in \operatorname{dom} M_2} \partial M_2(x) = \operatorname{range}(\partial M_2)$ define

the <u>exact conjugate of M_2 at L</u> as

$$M_2^*(L) = \max \{ L(x) - z : (x, z) \in \mathcal{G}(M_2) \}$$

Facts $\quad M_2^* : \operatorname{dom} M_2^* = \operatorname{range}(\partial M_2) \longrightarrow Z_2$ is a map

$$L(x) - z \leq M_2^*(L) \quad \forall L \in \operatorname{dom} M_2^*, \ \forall (x, z) \in \mathcal{G}(M_2)$$

$L \in \partial M_2(\bar{x}) \iff \min M_2(\bar{x})$ exists and $L(\bar{x}) - \min M_2(\bar{x}) = M_2^*(L)$

$(\gamma)\quad \exists x \in D : M_1(x) \cap \operatorname{int}(-S_1) \neq \phi \,, \quad M_2(x) \subset S_2$

Pr 2 Assume (γ) is consistent and M_2 is S_2-subdifferentiable on D. Then there exists $L \in \operatorname{dom} M_2^*$ such that (γ_L) is true, where

$(\gamma_L)\ \exists x \in D : M_1(x) \cap \operatorname{int}(-S_1) \neq \phi \,, \quad M_2^*(L) - L(x) \in -S_2 .$

We know an alternative formulation for (γ_L), namely,

$(\gamma'_L)\ \exists \lambda_1 \in S_1^+ \setminus \{0\}, \exists \lambda_2 \in S_2^+ : \langle \lambda_1, z_1 \rangle \geq \langle \lambda_2, L(x) - M_2^*(L) \rangle, \forall x \in D, \forall z_1 \in M_1(x)$

A possible alternative for (γ):

$(\gamma')\qquad$ " (γ'_L) is true for any $L \in \operatorname{dom} M_2^*$ "

One has in fact

Pr 3 Assume M_2 is S_2-subdifferentiable on D. Then,

(γ') is true $\Rightarrow$ (γ) is not true

For each $L \in \operatorname{dom} M_2^*$ let us introduce

$I(L)$ $\quad \exists a_L \in Z_1 : (a_L, 0) \in \operatorname{int}\left(S_1 \times S_2 + \bigcup_{x \in D} M_1(x) \times \{M_2^*(L) - L(x)\}\right)$

which is in particular satisfied whenever $\operatorname{int} S_1 \neq \emptyset$ and

$I_0(L)$ $\quad \exists x_L \in D : M_2^*(L) - L(x_L) \in \operatorname{int}(-S_2)$

Observe that $I_0(L) \Rightarrow M_2(x_L) \subset \operatorname{int} S_2$

<u>TH2</u> Let $M_i : \operatorname{dom} M_i \subset X \rightrightarrows Z_i$ convex, $X \supset C$ convex, S_i convex cone in Z_i, $i=1,2$, such that

$$\operatorname{int} S_1 \neq \emptyset \qquad S_2 \cap (-S_2) = \{0\}.$$

Assume $I(L)$ (or $I_0(L)$) holds for any $L \in \operatorname{dom} M_2^*$ and M_2 is S_2-subdifferentiable on $D = C \cap \operatorname{dom} M_1 \cap \operatorname{dom} M_2$.

Then precisely one of the following statements is true:

(γ) $\quad \exists x \in D : M_1(x) \cap \operatorname{int}(-S_1) \neq \emptyset,\ M_2(x) \subset S_2$

(γ') $\quad \forall L \in \operatorname{dom} M_2^*,\ \exists \lambda_1 \in S_1^+ \setminus \{0\},\ \exists \lambda_2 \in S_2^+ : \langle \lambda_1, z_1 \rangle \geq \langle \lambda_2, L(x) - M_2^*(L) \rangle,\ \forall x \in D,\ \forall z_1 \in M_1(x)$

The univoque case : $M_i(x) = \{H_i(x)\}$ $i = 1, 2$

Exact conjugate H_2^* of H_2 : $H_2^*(L) = \max\limits_{x \in \operatorname{dom} H_2} \left(L(x) - H_2(x)\right)$, $\forall L \in \operatorname{range}(\partial H_2)$

$$L \in \partial H_2(x) \Leftrightarrow H_2^*(L) = L(x) - H_2(x)$$

Let us introduce for each $L \in \operatorname{dom} H_2^*$:

$J(L)$ $\exists a_L \in Z_1 : (a_L, 0) \in \operatorname{int}\left(S_1 \times S_2 + \bigcup\limits_{x \in D} \{H_1(x)\} \times \{H_2^*(L) - L(x)\}\right)$

that holds in particular if

$J_0(L)$ $\exists x_L \in D : H_2^*(L) - L(x_L) \in \operatorname{int}(-S_2)$

Cor 3 Assume H_i is S_i-convex $(i = 1, 2)$, H_2 is S_2-subdifferentiable on $D = C \cap \operatorname{dom} H_1 \cap \operatorname{dom} H_2$, and $J(L)$ holds for any $L \in \operatorname{dom} H_2^*$.
Then precisely one of the following statements holds :

$\exists x \in D : H_1(x) \in \operatorname{int}(-S_1)$, $H_2(x) \in S_2$

$\forall L \in \operatorname{dom} H_2^*$, $\exists \lambda_1 \in S_1^+ \setminus \{0\}$, $\exists \lambda_2 \in S_2^+ : (\lambda_1 \circ H_1)(x) \geq \langle \lambda_2, L(x) - H_2^*(L) \rangle$, $\forall x \in D$

Example <u>Systems of mixed convex \ concave inequalities</u>

X t.v.s., $X \supset$ convex

$$f_1, \dots, f_p, g_1, \dots, g_q : X \to \mathbb{R} \cup \{+\infty\} \text{ convex}$$

$$F = \bigcap_{1 \le i \le p} \operatorname{dom} f_i \qquad G = \bigcap_{1 \le j \le q} \operatorname{dom} g_q \qquad G' = \bigcup_{x \in G} \partial g_1(x) \times \dots \times \partial g_q(x)$$

Assume $g_1, \dots, g_q$ are subdifferentiable on $D = C \cap F \cap G$ and

$$\forall \mu = (\mu_1, \dots, \mu_q) \in G', \ \exists x_\mu \in D : g_j^*(\mu_j) - \langle \mu_j, x_\mu \rangle < 0, \ \forall j = 1, \dots, q$$

$$\left(\Rightarrow g_j(x_\mu) > 0, \ \forall j = 1, \dots, q \right)$$

Then precisely one of the following statements is true

$$\exists x \in D : f_1(x) < 0, \dots, f_p(x) < 0, \ g_1(x) \geq 0, \dots, g_q(x) \geq 0$$

$$\begin{cases} \forall \mu \in G', \ \exists (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p_+ \setminus \{0\}, \ \exists (\beta_1, \dots, \beta_q) \in \mathbb{R}^q_+ \text{ such that:} \\ \sum_{i=1}^p \alpha_i f_i(x) \geq \sum_{j=1}^q \beta_j \left(\langle \mu_j, x \rangle - g_j^*(\mu_j) \right), \ \forall x \in D \end{cases}$$