Large Congestion Games: Wardrop or Poisson?

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Based on joint work with:

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You are planning your commute route for tomorrow.

Not sure about your exact departure time, nor who might be on the road.

A congestion game with a random set of players !

Congestion games model strategic interactions under crowding externalities.

Games with *"many small players"* are frequently modeled as nonatomic games with a continuum of players.

In which sense is a continuous model close to the discrete system ?

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The answer depends on what we mean by *"small players"*...

- player i has a small load $w_i \approx 0$ to be transported with certainty
- player i has a unit load but is present with small probability $p_i \approx 0$

Each interpretation yields a different continuous limit.

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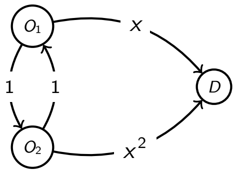
Network Congestion Games — $\mathscr{G}$

We are given a graph (V, E) with

- a set of *edges* $e \in E$ with continuous non-decreasing costs $c_e : \mathbb{R}_+ \to \mathbb{R}_+$
- a set of *OD pairs* $t \in T$ with corresponding routes $r \in \mathscr{R}_t \subseteq 2^E$
- a set of *demands* $d_t \geq 0$ for each $t \in T$



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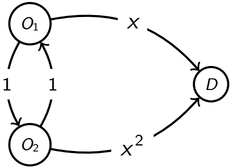

Demands can be...

- non-atomic: continuous, infinitesimal players $\to$ urban traffic
- atomic
  - splittable: continuous, few players $\to$ fluids, sand, telecom
  - unsplittable: discrete, few players $\to$ vessels, airplanes
  - random: unpredictable $\to$ packets or vehicles on a network

Non-Atomic Congestion Games — $\mathscr{G}^{na}$

- Continuum of players / each one has a negligible impact on congestion.
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Let $\mathscr{F}$ be the set of splittings (y, x) of the demands d_t into *route-flows* $y_r \geq 0$, together with their induced *edge-loads* x_e :

$$d_t = \sum_{r \in \mathscr{R}_t} y_r \quad (\forall t \in T),$$

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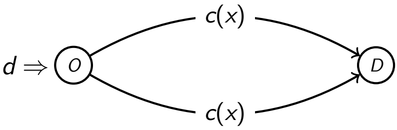
A Wardrop equilibrium *is a pair* $(\hat{y}, \hat{x}) \in \mathscr{F}$ *that uses only shortest routes:*

$$(\forall t \in T)(\forall r, r' \in \mathscr{R}_t) \quad \hat{y}_r > 0 \Rightarrow \sum_{e \in r} c_e(\hat{x}_e) \leq \sum_{e \in r'} c_e(\hat{x}_e).$$

Example: Single OD with 2 identical parallel links.



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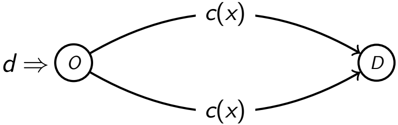

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Variational characterization:

Theorem (Beckmann-McGuire-Winsten, 1955)

Wardrop equilibria are exactly the optimal solutions of the convex program

$$\min_{(y,x)\in\mathscr{F}} \sum_{e\in E} \int_0^{x_e} c_e(z)\, dz.$$

Atomic Splittable Congestion Games — $\mathscr{G}^s$

Atomic splittable congestion games are similar in that demands are continuous and can be split over different routes, except that now:

1. There are finitely many players, each one controls a fraction of the demand.
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In what follows we address the discrete cases: unsplittable and random demands.

Weighted Congestion Games — $\mathscr{G}(w)$

A *weighted congestion game* has a finite set of players $i \in N$ with OD pairs $t_i \in T$, and unsplittable weights $w_i > 0$ that must be routed over a single path $r_i \in \mathscr{R}_{t_i}$ chosen at random using a mixed strategy $\pi_i \in \Delta(\mathscr{R}_{t_i})$.

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$$\mathbb{E}\left[\sum_{e \in r} c_e(X_e) | r_i = r\right] \leq \mathbb{E}\left[\sum_{e \in r'} c_e(X_e) | r_i = r'\right]$$

- WCGs with identical weights $w_i \equiv w$ are potential games and admit pure equilibria (Rosenthal'73). The potential for a profile $\mathbf{r} = (r_i)_{i\in N}$ is

$$\Phi(\mathbf{r}) = \sum_{e\in E} \sum_{k=1}^{n_e(\mathbf{r})} c_e(k\,w)\,w \qquad ; \qquad n_e(\mathbf{r}) \triangleq |\{i \in N : e \in r_i\}|.$$

- For heterogeneous weights we only have the existence of mixed equilibria.

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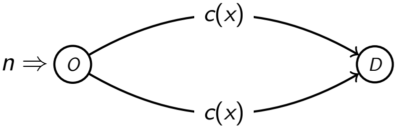

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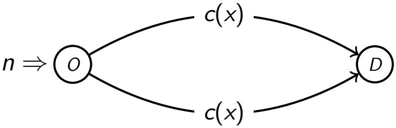

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If players' weights are $w_i \equiv d/n$ then we have random edge-loads

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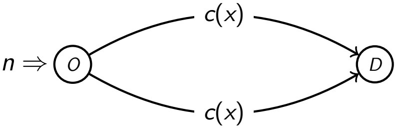

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What happens for other non-symmetric equilibria? What if weights are not homogeneous? And with different costs? And more complex topologies?

Wardrop Convergence for Vanishing Weights

Let π^n be a sequence of mixed equilibria for weighted ACGs $\mathscr{G}(w^n)$ with

$$\begin{cases} a) & |N^n| \to \infty \\ b) & \max_{i\in N^n} w_i^n \to 0 \\ c) & d_t^n \triangleq \sum_{i:t_i^n=t} w_i^n \to d_t \quad \text{for all } t \in T \end{cases}$$

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4. *If $c_e \in C^2$ with $c_e'(\cdot) > 0$, then there is a constant κ such that*

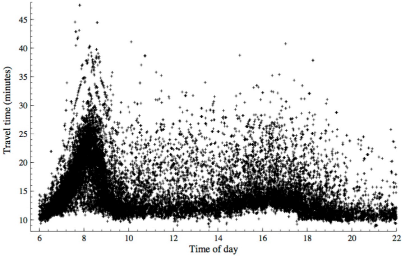
$$\|X^n - \hat{x}\|_{L^2} \leq \kappa\left(\sqrt{\max_{i \in N} w_i^n} + \sqrt{\|d^n - d\|_1}\right).$$

Simple and expected... but reality looks more like this

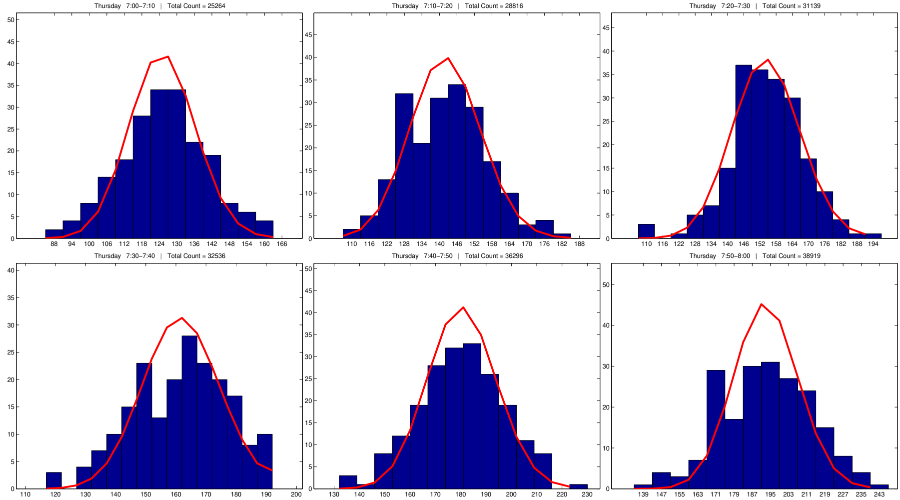
Copenhagen – *Source: DTU Transport (www.transport.dtu.dk)*

Figure 2: Example of real time illustration of congestion (Source: Vejdirektoratet, www.trafikken.dk)

Figure 7: Observations of travel time by time of day. Frederiks-sundsvej, inward direction



Traffic count data – Dublin 2017-2018



Data source: Transport Infrastructure Ireland

https://www.tii.ie/roads-tolling/operations-and-maintenance/traffic-count-data/

Bernoulli Congestion Games — $\mathscr{G}(p)$

A *Bernoulli congestion game* has a finite set of players $i \in N$ with OD pairs $t_i \in T$, unit weights $w_i = 1$, and a probability of being active

$$p_i = \mathbb{P}(U_i = 1).$$

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Remark. Costs need only be defined over the integers $c_e : \mathbb{N} \to \mathbb{R}_+$.

Bernoulli ACGs are Potential Games

Proposition

Every Bernoulli ACG is a potential game with potential

$$\Phi(\mathbf{r}) \triangleq \mathbb{E}\left[\sum_{e\in E}\sum_{k=1}^{N_e(\mathbf{r})} c_e(k)\right] \quad ; \quad N_e(\mathbf{r}) \triangleq \sum_{i:e\in r_i} U_i$$

Corollary

Every Bernoulli ACG has Nash equilibria in pure strategies.

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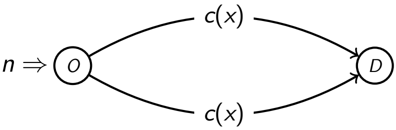

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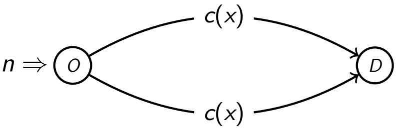

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If each player is present with probability $p_i = d/n$, the random edge-loads are

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which for large n converges to a $\mathrm{Poisson}(\frac{d}{2})$.

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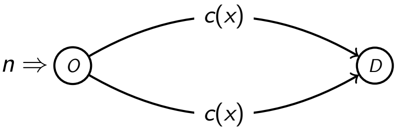

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What happens for other non-symmetric equilibria? What if players are not homogeneous? And with different costs? And more complex topologies?

Toolkit — Sums of Bernoullis ≈ Poisson

The total variation distance between two integer-valued random variables U, V is

$$d_{\mathrm{TV}}(U,V) = \tfrac{1}{2}\sum_{k\in\mathbb{N}} \big|\mathbb{P}(U=k) - \mathbb{P}(V=k)\big|.$$

Theorem (Barbour & Hall 1984, Borizov & Ruzankin 2002))

Let $S = X_1 + \ldots + X_n$ be a sum of independent Bernoullis with $\mathbb{P}(X_i = 1) \le p$, and $X \sim \mathrm{Poisson}(x)$ with the same expectation $\mathbb{E}[X] = x = \mathbb{E}[S]$. Then

$$d_{TV}(S,X) \le p.$$

Moreover, if $h : \mathbb{N} \to \mathbb{R}$ is such that $\mathbb{E}|\Delta^2 h(X)| \le \nu$, then

$$|\mathbb{E}h(S) - \mathbb{E}h(X)| \le \frac{x\,\nu}{2}\,\frac{p\,e^p}{(1-p)^2}.$$

REMARK: $\Delta^2 h(x) \triangleq h(x+2) - 2h(x+1) + h(x)$.

Poisson Convergence for Vanishing Probabilities

Standing Assumption: $\mathbb{E}[X^2 c_e(1+X)] < \infty$ for all $e \in E$ and $X \sim \mathrm{Poisson}(x)$.

This is a mild condition. It holds for costs with polynomial or exponential growth. It fails for fast growing costs such as factorials $k!$ or bi-exponentials $\exp(\exp(k))$.

We introduce the expected cost functions $\tilde{c}_e : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\boxed{\tilde{c}_e(x) \triangleq \mathbb{E}[c_e(1+X)] = \sum_{k=0}^{\infty} c_e(1+k) e^{-x} \frac{x^k}{k!}.}$$

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2. *Along any convergent subsequence we have*
   - *the edge-loads* X_e^n *converge in total variation to* $X_e \sim \text{Poisson}(\tilde{x}_e)$,
   - *the route-flows* Y_r^n *converge in total variation to* $Y_r \sim \text{Poisson}(\tilde{y}_r)$,
   - *the Poisson limits* Y_r *are independent.*

Poisson convergence for vanishing probabilities

Corollary

If the costs $c_e(k)$ are non-decreasing and non-constant, then the $\tilde{c}_e(\cdot)$'s are strictly increasing, the edge-loads $\tilde{x}_e$ are the same in all Wardrop equilibria, and for every sequence π^n of Bayes-Nash equilibria we have

$$X_e^n \xrightarrow{\mathrm{TV}} X_e \sim Poisson(\tilde{x}_e).$$

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Theorem

If $c_e(2) > c_e(1)$ for all $e \in E$ then there is a constant κ such that

$$d_{TV}(X_e^n, X_e) \leq \kappa\left(\sqrt{\max_{i\in N} p_i^n} + \sqrt{\|d^n - d\|_1}\right).$$

Summary and Comments

1. Both $w_i^n \to 0$ and $p_i^n \to 0$ lead to different non-atomic limit games:
   - For vanishing weights, the random edge-loads X_e^n converge in L^2 to the constants edge-loads $\hat{x}_e$.
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4. Poisson games were defined without reference to a limit process, so our convergence result as well as the connection with Wardrop seem new.

Convergence of PoA along sequences of ACGs

For an atomic congestion game $\mathscr{G}$ we denote

$$C(\pi) = \mathbb{E}_\pi\big[\textstyle\sum_{e\in E} X_e\, c_e(X_e)\big] \qquad \text{(expected social cost)}$$

$$C_{opt}(\mathscr{G}) = \min_\pi C(\pi) \qquad \text{(minimum social cost)}$$

$$\mathrm{PoA}(\mathscr{G}) = \max_{\pi\in\mathscr{E}(\mathscr{G})} C(\pi)/C_{opt}(\mathscr{G}) \qquad \text{(price-of-anarchy)}$$

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Theorem

Under the same conditions of the convergence theorems for weighted and Bernoulli ACGs, we have

$$\mathrm{PoA}(\mathscr{G}(w^n)) \longrightarrow \mathrm{PoA}(\mathit{Wardrop})$$

$$\mathrm{PoA}(\mathscr{G}(p^n)) \longrightarrow \mathrm{PoA}(\mathit{Poisson})$$

PoA for Bernoulli ACGs — Homogeneous players

How does PoA behaves as a function of the probabilities p_i ?

Theorem

Let $\mathscr{G}^p$ denote the set of Bernoulli ACGs with $p_i \leq p$ for all players. The largest values of $\mathrm{PoA}(\mathscr{G}(p))$ occur for homogeneous players with $p_i \equiv p$.

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From now on we focus on the homogeneous case and study PoA and PoS as a function of p when we move from the deterministic case $p = 1$ to the limit $p \downarrow 0$.

$$\mathrm{PoA}(p) = \sup_{\mathscr{G}^p} \max_{\pi \in \mathscr{E}(\mathscr{G}^p)} C(\pi)/C_{opt}(\mathscr{G}^p) \quad \text{(Price-of-Anarchy)}$$

$$\mathrm{PoS}(p) = \sup_{\mathscr{G}^p} \min_{\pi \in \mathscr{E}(\mathscr{G}^p)} C(\pi)/C_{opt}(\mathscr{G}^p) \quad \text{(Price-of-Stability)}$$

Related Literature

- Related models
  - Non-atomic with stochastic demand (Wang, Doan and Chen, 2014; Correa, Hoeksma and Schröder, 2019)
  - Smoothness with incomplete information (Roughgarden, 2015)
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- PoA for congestion games with affine costs
  - $\mathrm{PoA}(\mathscr{G}) \leq \frac{4}{3}$ for non-atomic (Roughgarden and Tardos, 2002)
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As a consequence of the latter we get $\mathrm{PoA}(p) \leq \frac{5}{2}$.

But we can find sharper bounds... and we expect $\mathrm{PoA}(p) \sim \frac{4}{3}$ for small p.

Smoothness Framework

Proposition

A Bernoulli ACG with homogeneous players is equivalent to a deterministic unweighted ACG for the auxiliary costs

$$c_e^p(k) = \mathbb{E}[c_e(1+B)] \textit{ with } B \sim \mathrm{Binomial}(k-1, p)$$

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Lemma (Roughgarden, 2015)

Let $\mathscr{G}$ be an unweighted ACG which is (λ, μ)-smooth with $\lambda > 0$ and $\mu \in (0,1)$, that is to say

$$(\forall s, s' \in S) \quad \textstyle\sum_{i \in N} C_i(s_i', s_{-i}) \le \lambda\, C(s') + \mu\, C(s).$$

Then we have $\mathrm{PoA}(\mathscr{G}) \le \frac{\lambda}{1-\mu}$.

Smoothness Framework — Affine Costs

Lemma

Let $\mathscr{P} = \{(k,m) \in \mathbb{N}^2 : k \geq 1\}$ *and suppose that* $\lambda > 0$ *and* $\mu \in (0,1)$ *satisfy*

$$k(1+pm) \leq \lambda\, k(1-p+pk) + \mu\, m(1-p+pm) \qquad \forall (k,m) \in \mathscr{P}. \tag{1}$$

Then every stochastic ACG $\mathscr{G}^p$ *with homogeneous players and affine costs is* (λ,μ)*-smooth, and therefore* $\mathrm{PoA}(p) \leq \frac{\lambda}{1-\mu}$.

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Let $\mathscr{P} = \{(k,m) \in \mathbb{N}^2 : k \geq 1\}$ *and suppose that* $\lambda > 0$ *and* $\mu \in (0,1)$ *satisfy*

$$k\,(1{+}pm) \leq \lambda\, k\,(1{-}p{+}pk) + \mu\, m\,(1{-}p{+}pm) \qquad \forall (k,m) \in \mathscr{P}. \tag{1}$$

Then every stochastic ACG $\mathscr{G}^p$ *with homogeneous players and affine costs is* (λ,μ)*-smooth, and therefore* $\mathrm{PoA}(p) \leq \frac{\lambda}{1-\mu}$.

The best combination of λ and μ for fixed p requires to solve

$$B(p) \triangleq \min_{\lambda>0,\mu\in(0,1)} \left\{ \frac{\lambda}{1-\mu} : \text{ subject to (1)} \right\}$$

which reduces to a 1D problem noting that the smallest λ compatible with (1) is

$$\lambda = \sup_{(k,m)\in\mathscr{P}} \frac{k(1{+}pm) - \mu\, m(1{-}p{+}pm)}{k(1{-}p{+}pk)}$$

Smoothness Framework — Affine Costs

The previous reduction leads to the equivalent minimization problem

$$B(p) = \inf_{\mu \in (0,1)} \varphi_p\left(\tfrac{\mu}{1-\mu}\right) = \inf_{y>0} \varphi_p(y)$$

where $\varphi_p(\cdot)$ is the convex envelop function

$$\varphi_p(y) = \sup_{(k,m)\in\mathscr{P}} \tfrac{1+pm}{1-p+pk} + \tfrac{k(1+pm)-m(1-p+pm)}{k(1-p+pk)}\, y.$$

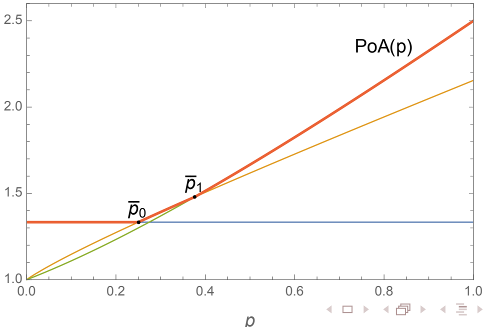
For each p the unique optimum y can be found explicitly, and then we recover the optimal combination (λ, μ).

Upper Bounds for the Price-of-Anarchy

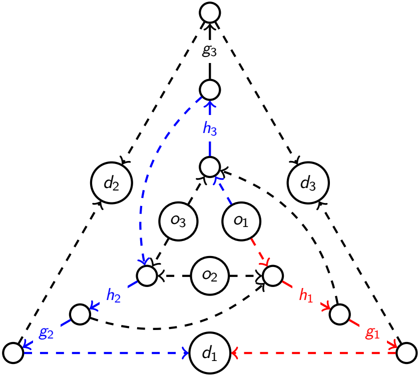
Theorem

Set $\bar{p}_0 = \frac{1}{4}$ and $\bar{p}_1 \sim 0.3774$ the unique real root of $8p^3 + 4p^2 = 1$. Then

$$\Rightarrow \qquad \mathrm{PoA}(p) \leq B(p) = \begin{cases} 4/3 & \text{if } 0 < p \leq \bar{p}_0, \\ \frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\ 1 + p + \frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1, \end{cases}$$



Lower Bounds for Large p



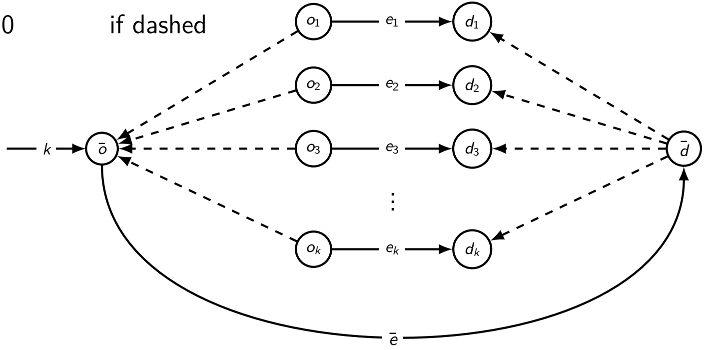
$$c_e(x) = \begin{cases} x & \text{if } e = h_i \\ p\,x & \text{if } e = g_i \\ 0 & \text{if dashed} \end{cases}$$

$$\Rightarrow \mathrm{PoA}(\mathscr{G}^p) = 1 + p + \frac{p^2}{1+p}.$$

Lower Bounds for Small p

$$c_e(x) = \begin{cases} \frac{1}{1+2kp}\, x & \text{if } e = \bar{e} \\ x & \text{if } e = e_i \\ 0 & \text{if dashed} \end{cases}$$

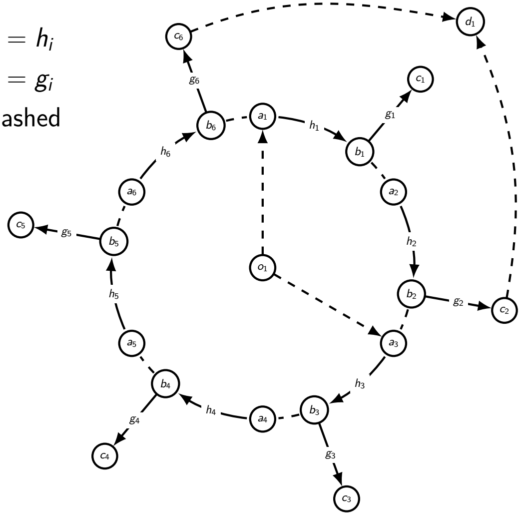
$2k$ players



$$\Rightarrow \mathrm{PoA}(\mathscr{G}^p) = \mathrm{PoS}(\mathscr{G}^p) \geq \frac{4kp+2-2p}{3kp+2-p} \to \frac{4}{3} \text{ as } k \to \infty$$

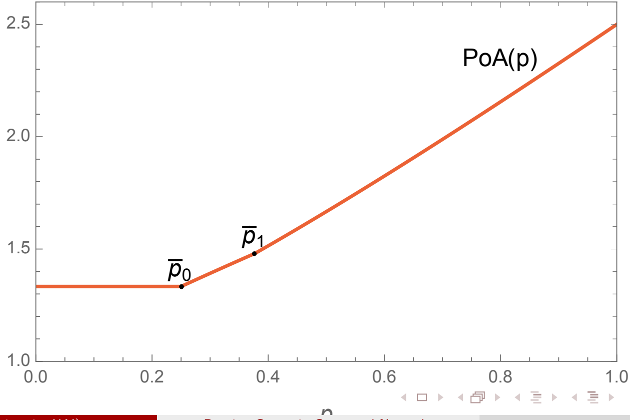
Lower Bounds for Intermediate p

$$c_e(x) = \begin{cases} \alpha\, x & \text{if } e = h_i \\ x & \text{if } e = g_i \\ 0 & \text{if dashed} \end{cases}$$



Bounds on the Price-of-Anarchy are Tight

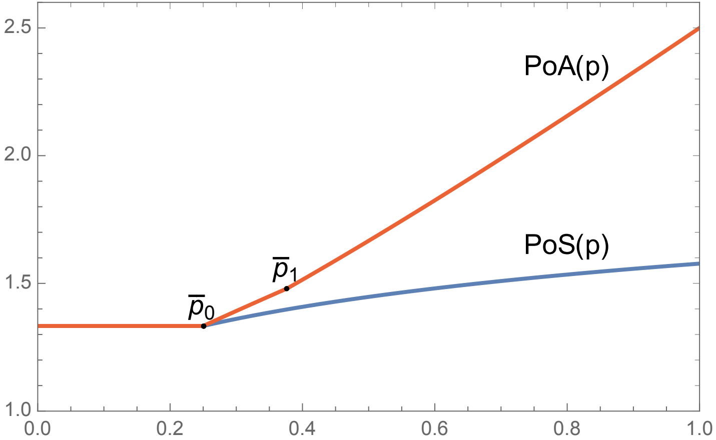
$$\mathrm{PoA}(p) = B(p) = \begin{cases} \frac{4}{3} & \text{if } 0 < p \leq \bar{p}_0 \\ \frac{1+p+\sqrt{p(2+p)}}{1-p+\sqrt{p(2+p)}} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1 \\ 1+p+\frac{p^2}{1+p} & \text{if } \bar{p}_1 \leq p \leq 1 \end{cases}$$



Price-of-Anarchy vs Price-of-Stability

Combining with Kleer and Schäfer (2018), we also get tight bounds for PoS

$$\mathrm{PoS}(p) = \begin{cases} 4/3 & \text{if } 0 < p \le \bar{p}_0 \\ 1 + \sqrt{p/(2+p)} & \text{if } p \ge \bar{p}_0 \end{cases}$$



Conclusion

1. Convergence of ACGs towards non-atomic games:
   - vanishing weights $\longrightarrow$ Wardrop
   - vanishing probabilities $\longrightarrow$ Poisson/Wardrop
2. Convergence of PoA/PoS, plus sharp bounds for affine costs
3. Some open questions
   - Mixed limits: weights & probabilities
   - Tight bounds for polynomial costs

Thanks !

Questions ?