On the approximation of first order mean field games

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Introduction: a static symmetric game

Let $Q \subseteq \mathbb{R}^d$ be compact and $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$ be continuous. We consider a game defined by

- N players.
- Set of actions Q (the same for all the players).
- A cost functional $F_i : Q^N \to \mathbb{R}$ for Player i defined by

$$F_i(x_1, \dots, x_i, \dots, x_N) = F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right).$$

Example: $N =$ number of swimmers, Q is a beach and

$$F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) = \alpha d(x_i, \text{snack bar}) - \frac{\beta}{N-1}\sum_{j\neq i}|x_i - x_j|$$

In order to obtain the existence of Nash equilibria, we consider the game with mixed strategies, i.e.

- The new set of actions is $\mathcal{P}(Q)$ (the same for all the players).
- The new cost functional $F_{rel,i} : \mathcal{P}(Q)^N \to \mathbb{R}$ for Player i defined by

$$F_{rel,i}(m_1, \ldots, m_i, \ldots, m_N) = \int_{Q^N} F_i(x_1, \ldots, x_i, \ldots, x_N) \otimes_{j=1}^N \mathrm{d}m_j(x_j).$$

A configuration $(\bar{m}_1, \ldots, \bar{m}_N)$ is a Nash equilibrium if $\forall\, i = 1, \ldots, N$

$$F_{rel,i}(\bar{m}_1, \ldots, \bar{m}_i, \ldots, \bar{m}_N) \leq F_{rel,i}(\bar{m}_1, \ldots, m, \ldots, \bar{m}_N) \quad \forall\, m \in \mathcal{P}(Q).$$

This relaxed framework and the symmetry of the game allow to show the existence of at least one equilibrium having the form

$$(\bar{m}^N, \ldots, \bar{m}^N) \quad \text{for some } \bar{m}^N \in \mathcal{P}(Q).$$

- The crucial point here is that, as $N \to \infty$, any limit point m^* of m_N satisfies

$$\int_Q F(x, m^*) \mathrm{d}m^*(x) = \min_{m \in \mathcal{P}(Q)} \int_Q F(x, m^*) \mathrm{d}m(x),$$

or, equivalently,

$$\text{supp}(m^*) \subseteq \text{argmin}\left\{F(x, m^*) \mid x \in Q\right\}.$$

Consequently, any of the two previous relations can be used to define the notion of MFG equilibrium.

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The dynamic and deterministic case (based on a joint work with M. Fischer)

A model problem in continuous time:

- Consider two continuous functions f, $g : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$, differentiable w.r.t. the first variable and satisfying that

$$\sup_{m \in \mathcal{P}_1(\mathbb{R}^d)} \{\|f(\cdot, m)\|_{\mathcal{C}^1} + \|g(\cdot, m)\|_{\mathcal{C}^1}\} \leq C,$$

- Consider N players, positioned at $x_1, \ldots, x_N \in \mathbb{R}^d$ at time $t = 0$.
- The set of actions for Player i is $\mathcal{A}(x_i)$, where

$$(\forall\, x \in \mathbb{R}^d) \;\; \mathcal{A}(x) := \left\{\gamma \in H^1([0,T]; \mathbb{R}^d) \;\middle|\; \gamma(0) = x\right\}.$$

- Given $\gamma_1 \in \mathcal{A}(x_1)$,..., $\gamma_N \in \mathcal{A}(x_N)$, the cost for Player i is

$$j_i(\gamma_1, \ldots, \gamma_i, \ldots, \gamma_N) = j\left(\gamma_i, \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_j}\right),$$

where, setting $\Gamma = C([0,T];\mathbb{R}^d)$, $j : H^1([0,T];\mathbb{R}^d) \times \mathcal{P}(\Gamma) \to \mathbb{R}$ is given by

$$j(\gamma_i, m) := \int_0^T \left[\tfrac{1}{2}|\dot{\gamma}_i(t)|^2 + f(\gamma_i(t), m(t))\right] \mathrm{d}t + g\left(\gamma_i(T), m(T)\right),$$

with

$$m(t) := e(t)\sharp m \quad \forall\, t \in [0,T].$$

Notice that

$$e(t)\sharp\left(\frac{1}{N-1}\sum_{j\neq i}\delta_{\gamma_j}\right) = \frac{1}{N-1}\sum_{j\neq i}\delta_{\gamma_j(t)}.$$

The previous game is not symmetric due to the heterogeneity of the initial conditions. In order to obtain a symmetric game let us

- assume that the initial positions of the players are randomly, identically and independently distributed.
- We denote by m_0 their common initial distribution, which is assumed to have a compact support.

- In this context, it is natural to define the action set of the players as

$$\mathcal{A} := \{\gamma : \mathbb{R}^d \to \Gamma \mid \gamma \text{ is Borel meas. and } \gamma^x := \gamma(x) \in \mathcal{A}(x), \ \forall\, x \in \operatorname{supp}(m_0)\}.$$

- Accordingly, given the strategies $\gamma_1 \in \mathcal{A}$, ..., $\gamma_N \in \mathcal{A}$, the cost functional for Player i is redefined as

$$J_i(\gamma_1, \ldots, \gamma_i, \ldots, \gamma_N) = J(\gamma_i, (\gamma_j)_{j \neq i}),$$

where

$$J(\gamma_i, (\gamma_j)_{j\neq i}) := \int_{(\mathbb{R}^d)^N} j\left(\gamma_i^{x_i}, \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_j^{x_j}}\right) \otimes_{j=1}^N \mathrm{d}m_0(x_j).$$

- Set $m_j := \gamma_j \sharp m_0$ (i.e. $\mathrm{d}m_j(\gamma) = \mathrm{d}\delta_{\gamma_j^x}(\gamma)\mathrm{d}m_0(x)$). Then

$$J(\gamma_i, (\gamma_j)_{j\neq i}) = \int_{\Gamma^N} j\left(\gamma_i', \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_j'}\right) \otimes_{j=1}^N \mathrm{d}m_j(\gamma_j').$$

- If $(\bar{\gamma}_1, \ldots, \bar{\gamma}_i, \ldots, \bar{\gamma}_N)$ is a Nash equilibrium for the previous game, then there exists $C > 0$, independent of N, such that

$$(\forall\, x \in \text{supp}(m_0)) \qquad \|\dot{\bar{\gamma}}^x\|_\infty \leq C.$$

- Set $Q_C := \{\gamma \in W^{1,\infty}([0,T];\mathbb{R}^d) \mid \|\dot{\gamma}\|_\infty \leq C, \;\; \gamma(0) \in \text{supp}(m_0)\}$.
- It is natural to consider as set of strategies the compact set

$$\mathcal{A}_{rel} := \{m \in \mathcal{P}(\Gamma) \mid e_0 \sharp m = m_0, \;\; \text{supp}(m) \subseteq Q_C\}.$$

and, as cost functional for Player i,

$$J_{rel,i}(m_1, \ldots, m_i, \ldots, m_N) = J_{rel}(m_i, (m_j)_{j \neq i}), \quad \text{with}$$

$$J_{rel}(m_i, (m_j)_{j \neq i}) = \int_{\Gamma^N} j\left(\gamma_i, \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_j}\right) \otimes_{j=1}^N \mathrm{d}m_j(\gamma_j).$$

- Then standard techniques show the existence of a Nash equilibrium for the previous game having the form $(\bar{m}^N, \dots, \bar{m}^N)$.

  Theorem: As $N \to \infty$, every limit point m^* of $(\bar{m}^N)_{N \in \mathbb{N}}$ satisfies

$$\int_\Gamma j(\gamma, m^*) \mathrm{d}m^*(\gamma) = \min_{m \in \mathcal{A}_{rel}} \int_\Gamma j(\gamma, m^*) \mathrm{d}m(\gamma),$$

  or, equivalently,

$$\text{supp}(m^*) \subseteq \{\gamma \in \Gamma \mid \gamma \in \text{argmin}\{j(\gamma', m^*) \mid \gamma'(0) = \gamma(0)\}\}. \qquad (*)$$

- A measure $m^* \in \mathcal{A}_{rel}$ is called a mean field game equilibrium if it satisfies $(*)$.

- The previous analysis shows, in particular, the existence of a MFG equilibrium.

- Suppose that in addition we have

$$\sup_{m \in \mathcal{P}_1(\mathbb{R}^d)} \{\|f(\cdot,m)\|_{\mathcal{C}^2} + \|g(\cdot,m)\|_{\mathcal{C}^2}\} \leq C,$$

  and that m_0 is absolutely continuous w.r.t. to $\mathcal{L}^d$.
- Then, associated to a MFG equilibrium m^*, there exists $\gamma^* \in \mathcal{A}$ such that $m^* = \gamma^* \sharp m_0$.
- Set $\rho(t) = m^*(t)$ and define the value function

$$v(t,x) = \inf \left\{ \int_t^T \left[\tfrac{1}{2}|\dot{\gamma}(s)|^2 + f(\gamma(s),\rho(s))\right] \mathrm{d}s + g\left(\gamma(T),\rho(T)\right) \ \middle| \ \gamma \in H^1([t,T];\mathbb{R}^d), \ \ \gamma(t) = x \right\}.$$

- Then the couple (v,ρ) solves

$$\left.\begin{array}{ll} -\partial_t v + \frac{1}{2}|\nabla v|^2 = f(x,\rho(t)), & v(\cdot,T) = g(\cdot,\rho(T)), \\ \partial_t \rho - \mathrm{div}(\nabla v \rho) = 0, & \rho(\cdot,0) = m_0. \end{array}\right\} \quad \text{(MFG)}$$

- Under the same assumptions, we also have the existence of $\bar{\gamma}^N \in \mathcal{A}$ such that $\bar{m}^N = \bar{\gamma}^N \sharp m_0$.

- As in the limit case, $\bar{m}^N$ can be characterized by the solution (v^N, ρ^N) of a PDE system similar system to (MFG).

- This allows a convergence proof based on PDE techniques.

- Uniqueness of a solution to (MFG) holds if, for $h = f$, g,

$$\int_Q (h(x,\mu) - h(x,\mu'))\ d(\mu - \mu')(x) \geq 0, \qquad \forall\, \mu,\ \mu' \in \mathcal{P}_1(\mathbb{R}^d).$$

- Related works:

  Existence of MFG equilibria in the deterministic case: Lasry-Lions '07, Cannarsa-Capuani '18, Cannarsa-Capuani-Cardaliaguet '18, Mazanti-Santambrogio '18, Achdou-Mannucci-Marchi-Tchou '19, Cannarsa-Mendico '19,...

  Convergence result: Lacker '16, Fischer '17, Cardaliaguet-Delarue-Lasry-Lions '19, Lacker '20, Gangbo-Mészáros '20,...

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Approximation of deterministic mean field games (based on a joint work with S. Hadikhanloo and an ongoing work with J. Gianatti)

Consider the MFG system

$$\left.\begin{array}{rcl} -\partial_t v + \frac{1}{2}|\nabla v|^2 & = & f(x,\rho(t)) \quad \text{in } [0,T]\times\mathbb{R}^d \\ v(T,\cdot) & = & g(\cdot,\rho(T)) \quad \text{in } \mathbb{R}^d \\ \partial_t \rho - \text{div}\big(\nabla v \rho\big) & = & 0 \quad \text{in } [0,T]\times\mathbb{R}^d \\ \rho(0,\cdot) & = & m_0 \quad \text{in } \mathbb{R}^d. \end{array}\right\} \qquad \text{(MFG)}$$

- A semi-Lagrangian scheme to solve (MFG) has been proposed in Carlini-S. '14. Full-convergence result when $d=1$.
- Fourier methods to treat (MFG) have been proposed recently in Nuberkyan-Saúde'19 and Li-Jacobs-Li-Nuberkyan-Osher '20.
- We describe now an fully-discrete scheme, which approximate general MFG equilibria in the form $(*)$.

- The discretization in Carlini-S.'14 is mainly based on the representation formulae

$$v(t,x) = \inf \int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + f(\gamma(s), m(s))\right] \mathrm{d}s + g(\gamma(T), m(T))$$

$$\text{s.t.} \qquad \dot{\gamma}(s) = \alpha(s) \quad \text{in } (t,T), \quad \gamma(t) = x$$

$$= \inf \int_t^T \left[\tfrac{1}{2}|\dot{\gamma}(s)|^2 + f(\gamma(s), m(s))\right] \mathrm{d}s + g(\gamma(T), m(T))$$

$$\text{s.t.} \qquad \gamma(t) = x,$$

and

$$\rho(t) = \gamma^{(\cdot)}(t)\sharp m_0,$$

where, for $x \in \mathbb{R}^d$, $\gamma^x(t)$ is the solution, evaluated at time t, to

$$\dot{\gamma}(s) = -\nabla v(s, \gamma(s)) \quad \text{in } (0,T), \quad \gamma(0) = x.$$

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Approximation of optimal control problem solved by the typical agent

At the equilibrium, a typical agent solves a problem having the form

$$\inf \int_0^T \left[\tfrac{1}{2}|\alpha(t)|^2 + f(\gamma(t))\right] \mathrm{d}t + g(\gamma(T)) \quad \text{s.t} \quad \dot{\gamma}(t) = \alpha(t), \;\; \gamma(0) = x.$$

The associated HJB equation is given by

$$\begin{aligned} -\partial_t v + \tfrac{1}{2}\left|\nabla v\right|^2 &= f \quad \text{in } [0,T]\times\mathbb{R}^d \\ v(T,\cdot) &= g \quad \text{in } \mathbb{R}^d \end{aligned}$$

- Let $\Delta t > 0$, set $t_k = k(\Delta t)$ and $\mathcal{T}_{\Delta t} := \{0, t_1, \ldots, t_n\}$, with $t_n = T$. A standard semi-discrete scheme to approximate v is

$$\begin{aligned} v_{\Delta t}(t_k, x) &= \inf_{\alpha\in\mathbb{R}^d} \left\{ \Delta t \left[\frac{|\alpha|^2}{2} + f(x) \right] + v_{\Delta t}(t_{k+1}, x + \Delta t \alpha) \right\}, \\ v_{\Delta t}(T, x) &= g(x). \end{aligned}$$

- Given a space-step $\Delta x > 0$, set $\mathcal{G}_{\Delta x} = \{x_i = i(\Delta x) \mid i \in \mathbb{Z}^d\}$. The fully-discrete SL scheme is

$$
\begin{aligned}
v_{\Delta t,\Delta x}(t_k, x_i) &= \inf_{\alpha \in \mathbb{R}^d} \left\{ \Delta t \left[\frac{|\alpha|^2}{2} + f(x_i) \right] + I[v_{\Delta t,\Delta x}](t_{k+1}, x_i + \Delta t \alpha) \right\}, \\
v_{\Delta t,\Delta x}(T, x_i) &= g(x_i),
\end{aligned}
$$

where $I[\cdot]$ is an interpolation operator associated to a triangulation with vertices in $\mathcal{G}_{\Delta x}$.

- Given the particular structure of the dynamics, we can avoid an infinite grid and, more importantly, interpolation by choosing controls such that

$$x_i + \Delta t \alpha \quad \text{is a grid point.}$$

Moreover, since the optimal control for the continuous problem will be bounded by some $C > 0$, it is natural to impose

$$\alpha = \frac{x_j - x_i}{\Delta t}, \qquad |\alpha| \leq C.$$

- Setting $\mathcal{G}_{\Delta x}(x_i) = \{x_j \in \mathcal{G}_{\Delta x} \mid |x_j - x_i| \leq C\Delta t\}$, we get

$$
\begin{aligned}
v_{\Delta t,\Delta x}(t_k, x_i) &= \inf_{x_j \in \mathcal{G}_{\Delta x}(x_i)} \left\{ \Delta t \left[\frac{1}{2} \left| \frac{x_j - x_i}{\Delta t} \right|^2 + f(x_i) \right] + v_{\Delta t,\Delta x}(t_{k+1}, x_j) \right\} \\
v_{\Delta t,\Delta x}(T, x_i) &= g(x_i),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
v_{\Delta t,\Delta x}(t_k, x_i) = \inf_{p \in \mathcal{P}(\mathcal{G}_{\Delta x}(x_i))} \Bigg\{ \sum_{x_j \in \mathcal{G}_{\Delta x}(x_i)} p_j \Bigg[\Delta t \left[\frac{1}{2} \left| \frac{x_j - x_i}{\Delta t} \right|^2 + f(x_i) \right] \\
+ v_{\Delta t,\Delta x}(t_{k+1}, x_j) \Bigg] \Bigg\},
\end{aligned}
$$

$$
v_{\Delta t,\Delta x}(T, x_i) = g(x_i).
$$

- For each (t_k, x_i) the problem defined by $v_{\Delta t,\Delta x}(t_k, x_i)$ can have several solutions. In order to get uniqueness, for $\varepsilon > 0$, consider the entropy penalized scheme

$$v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i) = \inf_{p\in\mathcal{P}(\mathcal{G}_{\Delta x}(x_i))}\left\{\sum_{x_j\in\mathcal{G}_{\Delta x}(x_i)} p_j\left[\Delta t\left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2 + f(x_i)\right]\right.\right.$$
$$\left.\left.+v^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j) + \varepsilon\log p_j\right]\right\},$$
$$v^{\varepsilon}_{\Delta t,\Delta x}(T,x_i) = g(x_i).$$

- For every t_k, x_i and $\varepsilon > 0$, the previous problem has a unique minimizer $p_{opt}(x_i,\cdot,t_k)$.

$p_{\mathrm{opt}}(x_i,x_j,t_k)$ denotes "optimal probability" of moving from x_i to x_j at time t_k.

- If $(\Delta t_n,\Delta x_n,\varepsilon_n)\to 0$, $\Delta x_n/\Delta t_n \to 0$, and $\varepsilon_n|\log(\Delta x_n)|/\Delta t_n \to 0$, then for every compact set $K\subseteq\mathbb{R}^d$ we have

$$\sup_{(t,x)\in\mathcal{T}_n\times(\mathcal{G}_{\Delta x_n}\cap K)}\left|v^{\varepsilon_n}_{\Delta t_n,\Delta x_n}(t,x) - v(t,x)\right| \underset{n\to\infty}{\to} 0.$$

- The previous scheme and the convergence result can be extended to several interesting contexts.

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Approximation of the MFG system

We consider the following discretization of the mean field game problem

$$v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i)=\inf_{p\in\mathcal{P}(\mathcal{G}_{\Delta x}(x_i))}\sum_{x_j\in\mathcal{G}_{\Delta x}(x_i)}p_j\left[c_{i,j}(p_j,m^{\varepsilon}_{\Delta t,\Delta x}(t_k,\cdot))+v^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j)\right]$$

$$v^{\varepsilon}_{\Delta t,\Delta x}(T,x_i)=g\left(x_i,m^{\varepsilon}_{\Delta t,\Delta x}(T,\cdot)\right)$$

$$m^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j)=\sum_{x_i\in\mathcal{G}_{\Delta x}}p_{\rm opt}(x_i,x_j,t_k)m^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i)$$

$$m^{\varepsilon}_{\Delta t,\Delta x}(0,x_i)=\tilde{m}_0(x_i)\qquad\forall\, x_i\in\mathcal{G}_{\Delta x},$$

where $\tilde{m}_0$ is a discretization of the initial distribution,

$$c_{i,j}(p_j,m)=\Delta t\left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2+f(x_i,m)\right]+\varepsilon\log(p_j),$$

$p_{\rm opt}(x_i,\cdot,t_k)\in\mathcal{P}(\mathcal{G}_{\Delta x}(x_i))$ is the unique minimizer of $v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i)$.

- The previous system is a particular instance of discrete time, finite state space MFGs introduced in Gomes-Mohr-Souza'10.
- Existence of a solution $(v^{\varepsilon}_{\Delta t,\Delta x}, m^{\varepsilon}_{\Delta t,\Delta x})$ follows from a fixed-point argument (see Gomes-Mohr-Souza'10).
- If f and g are monotone, i.e. for $h = f$, g,

$$\sum_{x_j \in \mathcal{G}_{\Delta x}} \left[h(x_j, m_1) - h(x_j, m_2)\right](m_1 - m_2) \geq 0 \quad \forall\, m_1, m_2 \in \mathcal{P}(\mathcal{G}_{\Delta x}),$$

  it is possible to show that the solution is unique.
- Under this monotonicity assumption we will show later how to compute the equilibrium $(v^{\varepsilon}_{\Delta t,\Delta x}, m^{\varepsilon}_{\Delta t,\Delta x})$ of (MFG_f).

- Consider a sequence $(\Delta t_n, \Delta x_n, \varepsilon_n) \to 0$, set

$$v^n = v^{\varepsilon_n}_{\Delta t_n, \Delta x_n}, \quad m^n = m^{\varepsilon_n}_{\Delta t_n, \Delta x_n},$$

and p^n_{opt} the optimal transition probabilities.

- Set $m^{*,n}$ for the probability measure on Γ induced by $\tilde{m}^n_0$ and p^n_{opt}.

The main result here is the following.

Theorem: Suppose that $\Delta x_n/\Delta t_n \to 0$, and $\varepsilon_n|\log(\Delta x_n)|/\Delta t_n \to 0$. Then the following holds

(i) Assume the "weak assumption" on the data f, g and m_0. Then, any limit point $m^* \in \mathcal{P}(\Gamma)$ of $(m^{*,n})_{n\in\mathbb{N}}$ is a mean field game equilibrium, i.e. it satisfies (*).

(ii) Assume the "strong assumption" on the data f, g and m_0. Then, if $m^{*,n} \to m^*$, we have that $(v^n, m^n) \to (v, \rho)$, where (v, ρ) is the solution to (MFG) associated to m^*.

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Solving the finite MFG problem

We consider the following "fictitious play" procedure to solve the discrete problem.

- Consider an arbitrary initial sequence of time marginals $M^1 = (M_0^1, \dots, M_N^1)$ and let $\bar{M}^1 = M^1$.
- For $\ell \geq 1$ compute

$$\begin{aligned} V_k^\ell &= \mathrm{HJB}(V_{k+1}^\ell, \bar{M}_k^\ell), \quad V_N^\ell = g(\bar{M}_N^\ell) \\ \text{and then} \quad M_{k+1}^{\ell+1} &= \mathrm{EV}(M_k^\ell, V_{k+1}^\ell), \quad M_0^{\ell+1} = m_0. \end{aligned}$$

  Set

$$\bar{M}^{\ell+1} := \frac{1}{\ell+1} \sum_{\ell'=1}^{\ell} M^{\ell'}.$$

- In terms of the best response (BR), the method can be written as

$$M^{\ell+1} = BR(\bar{M}^\ell).$$

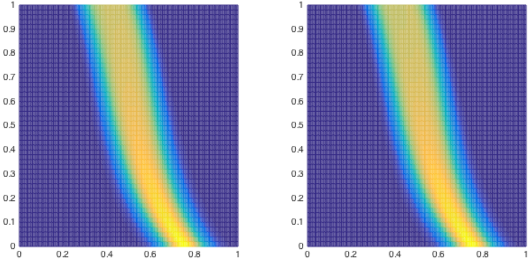
Theorem: [Hadikhanloo-S'19] If f and g are monotone and Lipschitz w.r.t. to the second argument, then $(V^\ell, M^\ell, \bar{M}^\ell) \to (v^n, m^n, m^n)$.

Example:

- Set $d = 1$, $T = 1$, $\rho_\sigma(z) = e^{-z^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, with $\sigma = 0.25$, and

$$
\begin{aligned}
f(x,m) &= 2(x-0.5)^2 + (\rho_\sigma * m) * \rho_\sigma(x) \\
g(x,m) &= 2(x-0.2)^2 + (\rho_\sigma * m) * \rho_\sigma(x) \\
m_0(x) &= \frac{h(x)}{\int_0^1 h(x')\mathrm{d}x'} \mathbb{I}_{[0,1]}(x), \quad \text{with } h(x) := e^{-\frac{(x-0.75)^2}{0.02}}
\end{aligned}
$$

- Discretization parameters: $\Delta x = 0.005$, $\Delta t = 0.02$ and $\varepsilon = 0.002$.
- We apply the fictitious play procedure to (MFG^f).



$\overline{M}^{\ell}$ (left) versus its best response $M^{\ell+1}$ (right), at step $\ell = 1000$.

- We have also tested the intuitive procedure $M^{\ell+1} = BR(M^{\ell})$. Convergence fails in general. Indeed, there are configurations M such that $M = BR(BR(M))$ and $M \neq BR(M)$.

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Extensions (ongoing work with J. Gianatti)

- The previous analysis can be adapted to dynamics having the form

$$\dot{\gamma}(t) = A(\gamma(t)) + B(\gamma(t))\alpha(t).$$

  For particular instances of the previous dynamics, the existence of MFG equilibria has been adressed in Cannarsa-Mendico'19 and Achdou-Mannucci-Marchi-Tchou'19.

- State constraints (Cannarsa-Capuani'18)

$$\gamma(t) \in K \quad \forall\, t \in [0, T].$$