Relationship between Pontryagin's principle and Hamilton-Jacobi approach for a class of control problems with intermediate state constraints.

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- Let $T>0$ be a given time horizon. For a given non-empty compact subset U of $\mathbb{R}^m$ ($m \geq 1$), define the set of admissible controls as:

$$\mathcal{U} := \Big\{ u : (0,T) \to \mathbb{R}^m, \text{ measurable, } u(t) \in U \text{ a.e } \Big\}.$$

- Consider the following control system:

$$(1) \qquad \begin{cases} \dot{y}(s) := f(y(s), u(s)), & \text{a.e } s \in [t,T] \\ y(t) := x, \end{cases}$$

where f is continuous on $\mathbb{R}^d \times U$ into $\mathbb{R}^d$, it is loc. Lipschitz continuous w.r.t y, and there exists $C>0$ s.t. $|f(y,u)| \leq C(1+|y|)$ for all $y \in \mathbb{R}^d$;

- For every $t \in [0,T]$, we define the set of trajectories:

$$S_{[t,T]}(x) := \Big\{ y_{t,x} \in W^{1,\infty}([t,T];\mathbb{R}^d), \;\; y_{t,x} \text{ satisfies (1) for some } u \in \mathcal{U} \Big\}.$$

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where f is continuous on $\mathbb{R}^d \times U$ into $\mathbb{R}^d$, it is loc. Lipschitz continuous w.r.t y, and there exists $C > 0$ s.t. $|f(y,u)| \leq C(1 + |y|)$ for all $y \in \mathbb{R}^d$;

Moreover, f is C^1 and $f(y, U)$ is a convex set for every $y \in \mathbb{R}^d$.

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The set of trajectories $\mathcal{S}_{[t,T]}(x)$ is a compact subset of $C^0([t,T]; \mathbb{R}^d)$.

Endpoint constraint (fixed time horizon T)

$$\begin{array}{ll} \min & \Phi(y(T)) \\ \text{s.t.} & y \in \mathcal{S}_{[t,T]}(x), \quad g_0(y(T)) \leq 0. \end{array}$$

- $g_0 : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function ;
- The final cost $\Phi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous (or C^1) function.

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Intermediate constraints

$$\begin{aligned} \min \quad & \Phi(y(T)) \\ \text{s.t.} \quad & y \in \mathcal{S}_{[t,T]}(x), \\ & g_1(y(t_1)) \le 0, \; g_0(y(T)) \le 0. \end{aligned}$$

- g_0 and g_1 are Lipschitz continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$; and t_1 is in $]0, T[$;
- The final cost $\Phi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous (or C^1) function.

Example (Goddard problem)

$$\dot r(s) = v(s)$$
$$\dot v(s) = \frac{1}{m}(F_T\, u(s) - F_D(r,v)) - \frac{1}{r(s)^2}$$
$$\dot m(s) = -\beta\, F_T\, u(s)$$
$$r(0)=1,\ v(0)=0,\ m(0)=1$$

$$y(\cdot)\left|\begin{array}{lll} r(\cdot) & : & \text{altitude} \\ v(\cdot) & : & \text{velocity} \\ m(\cdot) & : & \textit{masse} \end{array}\right.$$

$u(\cdot)$: % of the thrust

Here $F_D(r,v) = 310\, v^2\, e^{-500\,(r-1)}$, $\beta = 2$, and $F_T = 3.5$

- The ratio $u(t)$ is subject to the following constraint: $0 \le u(t) \le 1$.
- The rocket's altitude satisfies the final constraint ($r_f^* = 1.01$): $r(T) \ge r^*$.

The optimal control problem is the following (for $T > 0$):

$$\left\|\begin{array}{l} \min\ -m(T) \\ u(s) \in [0,1] \ \text{ a. e.,} \\ g_0(y(T)) := r^* - r(T) \le 0. \end{array}\right.$$

Trajectory optimization problem for a space launcher

Aim

Maximize the payload m_0 to be steered from the Earth (Kourou) to a prescribed Orbit (SSO, GEO, ...).

Collaboration with E. Bourgeois (CNES)

- The physical model involves 6+1 state variables, the position $\overrightarrow{X}$ of the launcher in the 3D space, its velocity $\overrightarrow{V}$ and its mass M:

$$\mathbf{y} := (\mathbf{X}, \mathbf{V}, \mathbf{M}).$$

- Forces acting on the rocket : Gravity $M\overrightarrow{g}$, Thrust $\overrightarrow{F_T}$, Drag $\overrightarrow{F_D}$, Coriolis $\overrightarrow{\Omega}$.

- Newton's Law:

$$\frac{d\overrightarrow{X}}{dt} = \overrightarrow{V},$$
$$\frac{d\overrightarrow{V}}{dt} = \overrightarrow{g} + \frac{\overrightarrow{F_T}}{M} + \frac{\overrightarrow{F_D}}{M} - 2\overrightarrow{\Omega} \wedge \overrightarrow{V} - \overrightarrow{\Omega} \wedge (\overrightarrow{\Omega} \wedge \overrightarrow{X}),$$

- The launcher is controlled by means of:
  - *launch parameters* $\Pi = (\psi, \omega) \in \mathbb{R}^2$
  - *incidence and sideslip angles* $\alpha(\cdot), \delta(\cdot)$

- Maximize the *payload*
- Physical state-constraints (intermediate times, end-point)

Outline

Outline

Pontryagin's Maximum Principle (PMP)

Mayer Problem without state constraints

$$\min\{\Phi(y(T)) \quad | \quad y \in \mathcal{S}_{[0,T]}(x_0)\}.$$

Optimality conditions - PMP

- Let $(\overline{y}, \overline{u})$ an optimal pair. There exists $\overline{p} \in W^{1,\infty}([0,T]; \mathbb{R}^d)$ satisfying:

$$\dot{\overline{y}}(s) = -\partial_p H(\overline{y}(s), \overline{p}(s), \overline{u}(s)), \qquad \overline{y}(0) = x_0,$$

$$\dot{\overline{p}}(s) = \partial_y H(\overline{y}(s), \overline{p}(s), \overline{u}(s)), \qquad \overline{p}(T) \in \partial\Phi(\overline{y}(T)),$$

$$H(\overline{y}(s), \overline{p}(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}(s), u), \quad \text{a.e. } s \in [0,T].$$

- where $H(x,p,u) := -\langle p, f(x,u) \rangle$

Ref: Boltyanskii'57, Pontryagin et al.'61-62, Warga'72, Sussman'94-98, Vinter'00, Clarke'13, Mordukhovich'06 ...

Hamilton-Jacobi-Bellman (HJB) approach

$$\vartheta(t,x) = \min_{y \in \mathcal{S}_{[t,T]}(x)} \Phi(y(T))$$

- ϑ satisfies the dynamic programming principle:

$$\begin{aligned} \vartheta(t,x) &= \min_{y \in \mathcal{S}_{[t,t+h]}} \vartheta(t+h, y(t+h)) \quad h \in (0, T-t). \\ \vartheta(T,x) &= \Phi(x) \end{aligned}$$

- ϑ is the unique bounded lsc (or continuous) *viscosity* solution of the HJB equation:

$$\begin{aligned} &-\partial_t \vartheta(t,x) + \max_{u \in U}(-D_x \vartheta(t,x) \cdot f(x,u)) = 0 \\ &\vartheta(T,x) = \Phi(x). \end{aligned}$$

Case Φ Lipschitz continuous: Bellman, Subbotina, Evans, P.L Lions, Cappuzo-Dolcetta/Bardi, Barles,...

Case Φ lsc: Barron-Jensen, Frankowska, ...
Souganidis, Barles, Soner, Ishii,, ...

Relationship between PMP and HJB approach

Mayer Problem (free of state constraints)

$$\vartheta(t,x) := \min\{\Phi(y(T)) \;\;|\;\; y \in \mathcal{S}_{[t,T]}(x)\}, \qquad x \in \mathbb{R}^d \text{ and } 0 \le t \le T.$$

Let $(\bar{y}, \bar{u})$ be an optimal solution for the problem with $t = 0$ and $x = y_0$ fixed.

- There exists $\overline{p} \in W^{1,\infty}([0,T];\mathbb{R}^d)$ satisfying:

$$\begin{cases} \dot{\overline{p}}(s) & = & \partial_y H(\overline{y}(s), \overline{p}(s), \overline{u}(s)), \\ \overline{p}(T) & \in & \partial\Phi(\overline{y}(T)). \end{cases}$$

$$H(\overline{y}(s), \overline{p}(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}(s), u) \;\; (=: \mathcal{H}(\overline{y}(s), \overline{p}(s))) \qquad \text{a.e.}$$

- The value function is the unique viscosity solution to the HJB equation:

$$-\partial_t \vartheta(t,x) + \mathcal{H}(x, D_x\vartheta(t,x)) = 0 \qquad x \in \mathbb{R}^d,\; t \in [0,T[,$$

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- The value function is the unique viscosity solution to the HJB equation:

$$-\partial_t\vartheta(t,x) + \mathcal{H}(x, D_x\vartheta(t,x)) = 0 \qquad x \in \mathbb{R}^d,\; t \in [0,T[,$$
$$\vartheta(T,x) = \Phi(x).$$

- Moreover, $\overline{p}(0) \in \partial_y\vartheta(0, y_0)$ & $\overline{p}(s) \in \partial_y\vartheta(s, \overline{y}(s))$ a.e. on $[0,T]$,

$$\Big(\mathcal{H}(\overline{y}(s), \overline{p}(s)), \overline{p}(s)\Big) \in \partial_{s,y}\vartheta(s, \overline{y}(s)) \;\text{ a.e. on } [0,T].$$

Ref: Clarke-Vinter'87, Vinter'87, Vinter'00, Bettiol-Vinter'10, Cristiani-Martinon'10, Bettiol-al.'16, ...)

Example (Clarke-Vinter'87):

$$\begin{array}{rl} \text{Minimize} & \Phi(y(1)) \\ \text{subject to} & \dot{y}(s) = y(s)u(s) \;\; s \in (0,1), \;\; y(0) = 0, \\ & u(s) \in [0,1] \;\; s \in [0,1]. \end{array}$$

Here, we consider

$$\Phi(x) := \begin{cases} -x & \text{if } x > 0, \\ -e^{1/2}x & \text{if } x \leq 0. \end{cases}$$

Then the value function is

$$\vartheta(s,y) = \begin{cases} -e^{1-s}y & \text{if} y > 0, \\ -e^{1/2}y & \text{if } y \leq 0. \end{cases}$$

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- The costate satisfies $\overline{p}(\cdot) \equiv \lambda \in \partial\Phi(\overline{y}(1))$.

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- The optimal trajectory is $\bar{y} \equiv 0$ and an optimal control is $\bar{u} \equiv 0$.
- The costate satisfies $\bar{p}(\cdot) \equiv \lambda \in \partial\Phi(\bar{y}(1))$.
- The sensitivity relation $\bar{p}(s) \in \partial_y\vartheta(s, \bar{y}(s))$ is satisfied only for $\lambda = -e^{1/2}$.

Outline

➤ Consider the endpoint constrained control problem

$$
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\vartheta(t,x) := \quad & \min \quad \Phi(y(T)) \\
& \text{s.t.} \quad y \in \mathcal{S}_{[t,T]}(x), \\
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- In general, the value function ϑ is merely l.s.c., and may take infinite values.
- **If** an optimal solution $(\overline{y}, \overline{u})$ exists, then the PMP holds with the adjoint state satisfying:

$$\begin{aligned} \dot{\overline{p}}(s) &= H'_y(\overline{y}(s), \overline{p}(s), \overline{u}(s)), \\ \overline{p}(T) &= \mu \nabla \Phi(\overline{y}(T)) + \lambda_0 \nabla g_0(\overline{y}(T)) \end{aligned}$$

where $(\mu, \lambda_0) \in \{0,1\} \times \mathbb{R}^+$, $(\mu, \lambda_0) \neq 0$ and $\min(\lambda_0, -g_0(\overline{y}(T))) = 0$.

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- At this stage, it is not clear how to generalize the link between the PMP and HJB since ϑ is only l.s.c

An auxiliary control problem (Altarovici-Bokanowski-HZ'13)

➤ Consider the following auxiliary control problem ($z \in \mathbb{R}$):

$$w(t,x,z) \quad := \quad \min_{y \in \mathcal{S}_{[t,T]}(x)} \left\{ (\Phi(y(T)) - z) \bigvee g_0(y(T)) \right\}$$

with $a \bigvee b = \max(a,b)$.

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- The function w is the unique continuous solution to:

$$-\partial_t w(t,x,z) + \max_{u \in U}(-D_x w(t,x,z) \cdot f(x,u)) = 0 \qquad \text{on } [0,T) \times \mathbb{R}^{d+1},$$

$$w(T,x,z) = (\Phi(x) - z) \bigvee g_0(x) \qquad \text{on } \mathbb{R}^{d+1}.$$

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Improvement function (Mifflin'77, Solodov-Sagastizábal'04, Apkarian-et-al.'08, ...)

$$(\mathcal{P}) \qquad \min_{G(X) \leq 0} F(X)$$

- The auxiliary optimization problem:

$$(\widehat{\mathcal{P}}) \qquad \min_X \left\{ (F(X) - z) \bigvee G(X) \right\}$$

- We have the equivalence

$$\bar{X} \text{ is optimal for } (P) \Leftrightarrow \begin{cases} \bar{X} \text{ solution of } (\widehat{\mathcal{P}}) \ \text{ for } \bar{z} = F(\bar{X}), \\ \min_X \left\{ (F(X) - \bar{z}) \bigvee G(X) \right\} = 0. \end{cases}$$

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with $a \bigvee b = \max(a,b)$.

For every $x \in \mathbb{R}^d$, we have:

(i) $\mathcal{E}\mathrm{pi}\ \vartheta(t,\cdot) = \Big\{ (x,z) : w(t,x,z) \le 0 \Big\}$,

(ii) $\vartheta(t,x) = \min \Big\{ z \in \mathbb{R} \,,\ w(t,x,z) \le 0 \Big\}$

(iii) Let $\bar{y}$ be an optimal solution of the auxiliary control problem with $\bar{z} = \vartheta(t,x)$, then $\bar{y}$ is also an optimal solution of the original endpoint constrained problem

Relationship between the PMP and HJB (Bokanowski-Desilles-HZ (hal-03079781))

- Let $\overline{y}$ be an optimal solution of the auxiliary control problem with $\overline{z} = \vartheta(0, y_0)$.

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- Let $\overline{y}$ be an optimal solution of the auxiliary control problem with $\overline{z} = \vartheta(0, y_0)$.
- There exists $(\overline{p}_x, \overline{p}_z)$ satisfying:

$$\begin{array}{rcl} \dot{\overline{p}}_x(s) & = & H'_y(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) \\ \dot{\overline{p}}_z(s) & = & 0 \\ \begin{pmatrix} \overline{p}_x(T) \\ \overline{p}_z(T) \end{pmatrix} & \in & \partial_{x,z} \left\{ (\Phi(\overline{y}(T)) - \overline{z}) \bigvee g_0(\overline{y}(T)) \right\} \end{array}$$

- For a.e $s \in [0, T]$,

$$H(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}_x(s), u).$$

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\overline{p}_z(T) &= -\mu
\end{aligned}
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where $(\mu, \lambda_0) \in \{0, 1\} \times \mathbb{R}^+$, $\mu + \lambda_0 = 1$, and $\min(\lambda_0, -g_0(\overline{y}(T))) = 0$.

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- For a.e $s \in [0, T]$,

$$H(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}_x(s), u).$$

- Moreover,

$$(\overline{p}_x(0), -\mu) \in \partial_{y,z} w(0, y_0, \overline{z}), \qquad (\overline{p}_x(s), -\mu) \in \partial_{y,z} w(s, \overline{y}(s), \overline{z}) \quad \text{a.e,}$$

and $$(\mathcal{H}(\overline{y}(s), \overline{p}_x(s)), -\overline{p}(s)) \in \partial_{s,y,z} w(s, \overline{y}(s), \overline{z})$$

- We first compute an approximation w^{Δ} of the auxiliary value function w.

- Compute an approximation of $\overline{p}^{\Delta}(0) = \nabla_x w^{\Delta}(0, x_0, \overline{z})$.

- Solve the optimality condition with the value of $\overline{p}^{\Delta}(0)$. We get an optimal solution (y^{Δ}, u^{Δ}) of the control problem.

- We use the relations

$$\begin{aligned} p_x^{\Delta}(T) &= \mu \nabla \Phi(y^{\Delta}(T)) + \lambda_0 \nabla g_0(y^{\Delta}(T)) \\ -\mu &= \nabla_z w^{\Delta}(0, x_0, \overline{z}) \end{aligned}$$

and get approximate values for (μ, λ_0).

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  Lipschitz viscosity notion provides a convenient framework for convergence analysis of monotone and consistent numerical schemes

- Compute an approximation of $\overline{p}^{\Delta}(0) = \nabla_x w^{\Delta}(0, x_0, \overline{z})$.
  Numerical computations of the gradient show a **stable** behavior (Benamou et al., JCM'10).

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- We use the relations

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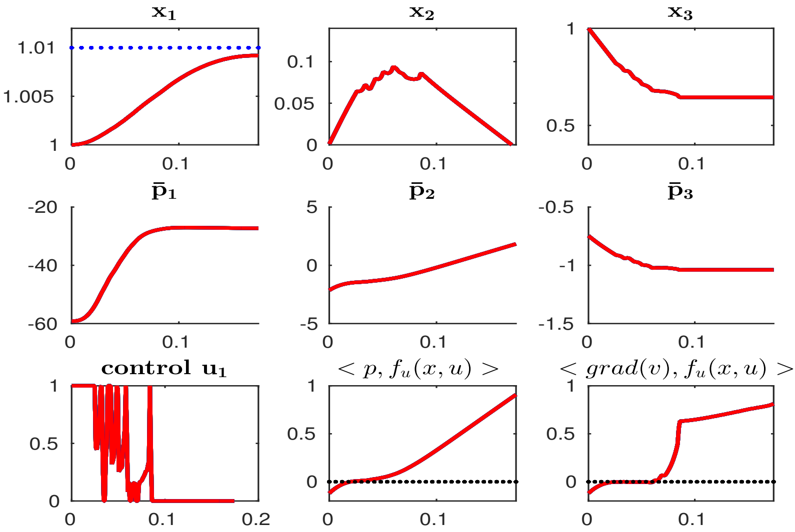
First approximation by using HJB equation on a coarse grid ($T = 0.174$)



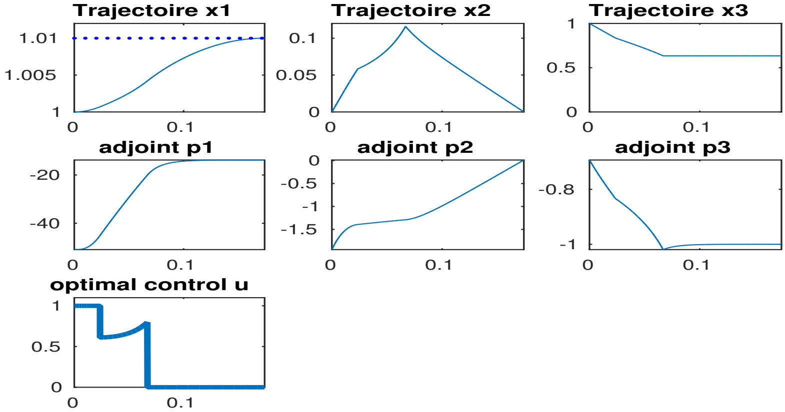


Figure: An optimal trajectory obtained by PMP principle combined with the HJB approach.

Outline

Intermediate constraints ($t_1 \in]0, T[$ is fixed)

$$\begin{aligned} \min \quad & \Phi(y(T)) \\ \text{s.t.} \quad & y \in \mathcal{S}_{[t,T]}(x), \\ & g_1(y(t_1)) \leq 0, \; g_0(y(T)) \leq 0. \end{aligned}$$

An auxiliary control problem (Altarovici-Bokanowski-HZ'13)

- Consider the following auxiliary control problem ($z \in \mathbb{R}$):

$$w(t,x,z) \quad := \quad \begin{cases} \inf\limits_{y \in \mathcal{S}_{[t,T]}(x)} \{(\Phi(y(T)) - z) \bigvee g_0(y(T))\} & \text{for } t \in]t_1, T] \\ \inf\limits_{y \in \mathcal{S}_{[t,T]}(x)} \{(\Phi(y(T)) - z) \bigvee g_0(y(T)) \bigvee g_1(y(t_1))\} & \text{for } t \in [0, t_1] \end{cases}$$

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- The function w is continuous on $[0, t_1[\times\mathbb{R}^{d+1}$ and on $]t_1, T] \times \mathbb{R}^{d+1}$. It is the unique solution of:

$$-\partial_t w(t,x,z) + \mathcal{H}(x, D_x w(t,x,z)) = 0 \quad \text{on }]t_1, T) \times \mathbb{R}^{d+1},$$
$$-\partial_t w(t,x,z) + \mathcal{H}(x, D_x w(t,x,z)) = 0 \quad \text{on } [0, t_1[) \times \mathbb{R}^{d+1},$$
$$w(T,x,z) = (\Phi(x) - z) \bigvee g_0(x) \quad \text{on } \mathbb{R}^{d+1},$$
$$w(t_1^-,x,z) = w(t_1^+,x,z) \bigvee g_1(x) \quad \text{on } \mathbb{R}^{d+1}.$$

Relationship between the PMP and HJB

- Let $\overline{y}$ be an optimal solution of the auxiliary control problem with $\overline{z} = \vartheta(t, y_0)$.
- There exists $(\overline{p}_x, \overline{p}_z)$ satisfying on $]0, t_1[\bigcup]t_1, T[$:

$$\begin{aligned} \dot{\overline{p}}_x(s) &\in H'_y(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) \\ \overline{p}_z(s) &= \mu \\ \overline{p}_x(T) &\in \mu \nabla \Phi(\overline{y}(T)) + \lambda_0 \partial g_0(\overline{y}(T)) \end{aligned}$$

$$\overline{p}_x(s)(t_1^-) - \overline{p}_x(s)(t_1^+) \in \lambda_1 \partial g_1(\overline{y}(t_1)),$$

where $(\mu, \lambda_0, \lambda_1) \in \{0, 1\} \times \mathbb{R}^+ \times \mathbb{R}^+, \quad \mu + \lambda_0 + \lambda_1 = 1$,
and $\min(\lambda_i, -g_i(y(t_i))) = 0$ for $i = 0, 1$.

- For a.e $s \in [0, T]$, $\quad H(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}_x(s), u)$.

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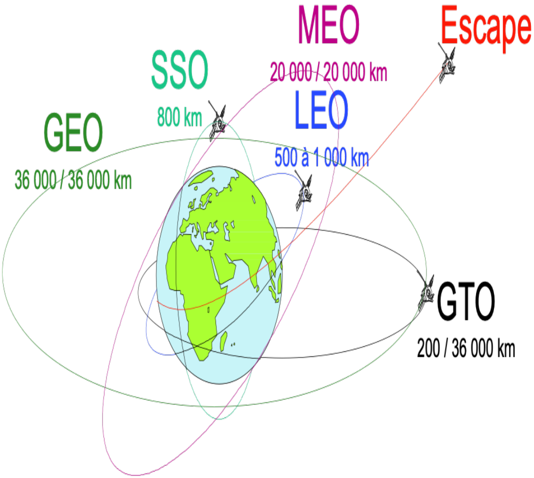
- For a.e $s \in [0, T]$, $\quad H(\overline{y}(s), \overline{p}_x(s), \overline{u}(s)) = \max_{u \in U} H(\overline{y}(s), \overline{p}_x(s), u)$.
- Moreover, on $]0, t_1[\bigcup[t_1^+, T[$, we have:

$$\overline{p}_x(0) \in \partial_y w(0, y_0, \overline{z}), \quad \overline{p}_x(t_1^{\pm}) \in \partial_y w(t_1^{\pm}, \overline{y}(t_1^{\pm}), \overline{z}),$$

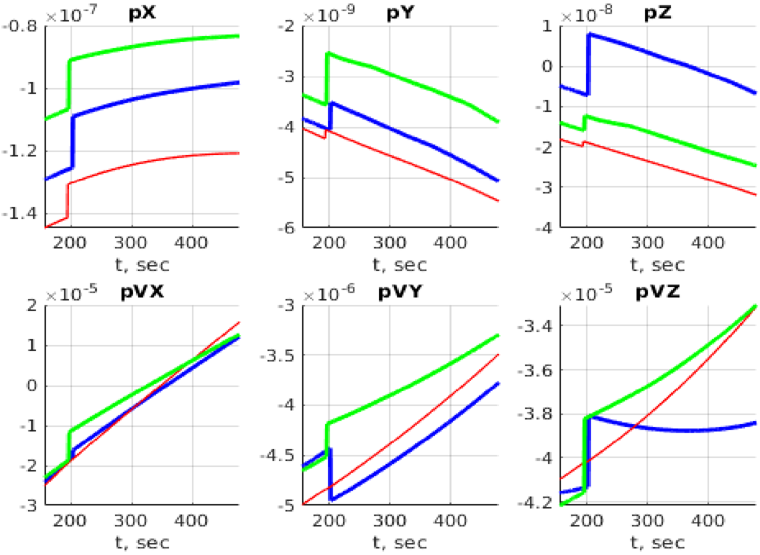
$$\overline{p}_x(s) \in \partial_y w(s, \overline{y}(s), \overline{z}), \qquad -\mu \in \partial_z w(0, \overline{y}(0), \overline{z}),$$

and $\quad (\mathcal{H}(\overline{y}(s), \overline{p}_x(s)), -\overline{p}(s)) \in \partial_{s,y,z} w(s, \overline{y}(s), \overline{z})$

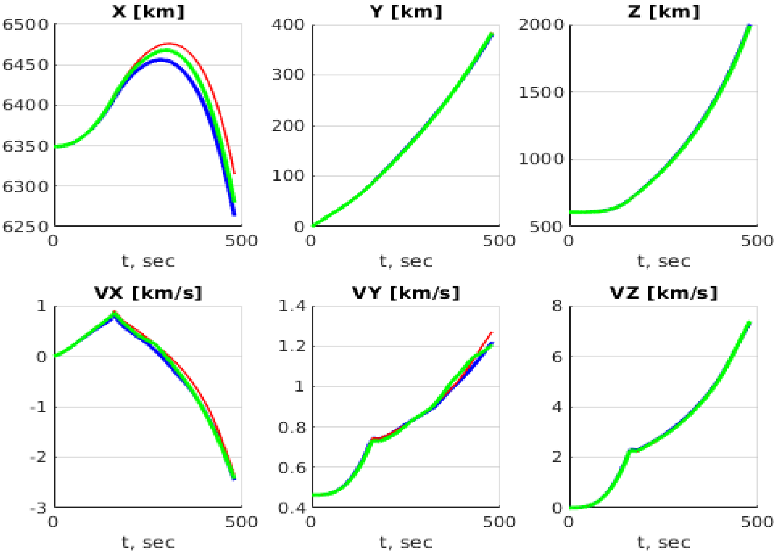
Trajectory optimization under final and intermediate constraints (software developed in coll. with CNES)



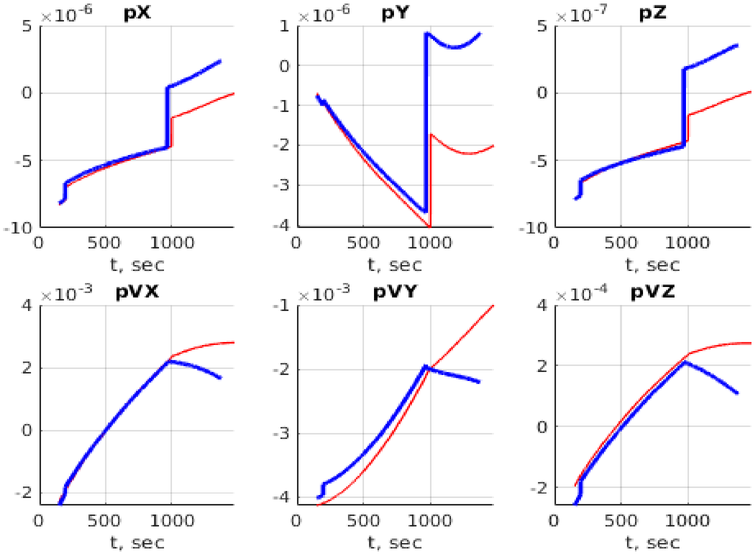
State adjoint (Example 1)



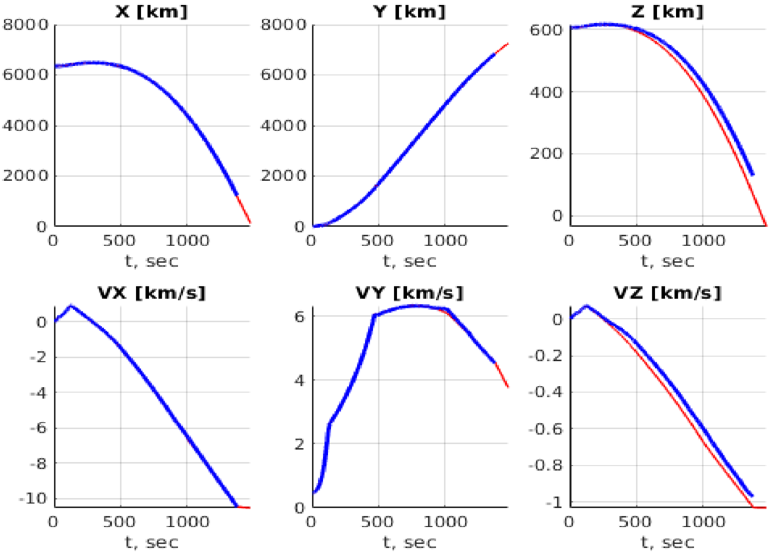
Optimal trajectory (Example 1)



State adjoint (Example 2)



Optimal trajectory (Example 2)



Conclusion

- When the optimal control problem is in presence of state constraints, the value function is merely l.s.c unless some strong controllability assumptions are satisfied.
- The main feature of introducing the auxiliary control problem is that its value function is more smooth, it can be easily characterized by a HJB equation.
- The link between the HJB approach and PMP can be established by using the auxiliary function (and not the original discontinuous value function).
- The original proof of Vinter'86 can be extended to prove existence of an adjoint state satisfying all the sensitivity relations.

....thanks for your attention.