Some aspects of MFG

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Mean Field Games (MFG) study collective behavior of rational agents.

- **collective behavior** = infinitely many agents, having individually a negligible influence on the global system
- **rational agents** = each agent controls his state in order to minimize a cost which depends on the other agents’ positions

Some references:


— Similar models in the economic literature: heterogeneous agent models (Aiyagari (‘94), Krusell-Smith (‘98),...)

— Recent monographs by Carmona-Delarue (‘18) and Achdou-C.-Delarue-Porretta-Santambrogio (‘20)
1. A short survey of MFG
2. MFG with common noise
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The $N$–player game

- **$N$ players**

- **Dynamics:** $dX^i_t = \alpha^i_t dt + dB^i_t + \beta dW_t$,
  (where the $B^i$ and $W$ are i.i.d. B.M., $\alpha^i$ is the control of Player $i$, and $X^i_0$ are i.i.d. of law $\bar{m}_0$)

- **Goal of the players:** to *minimize* over $\alpha^i$ the cost

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E} \left[ \int_0^T L(X^i_t, \alpha^i_t, m^{N,i}_X) dt + G(X^i_T, m^{N,i}_X) \right],$$

where $m^{N,i}_x = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$ if $x = (x_1, \ldots, x_N)$.

- **Nash equilibrium:** $(\bar{\alpha}^1, \ldots, \bar{\alpha}^N)$ s.t., for any $i \in \{1, \ldots, N\}$, $\bar{\alpha}^i$ minimizes $\alpha^i \to J^i(\alpha^i, (\bar{\alpha}^j)_{j \neq i})$. 
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The formal mean field limit game \((N \to +\infty)\): limit of dynamics

Assume that in the \(N\)–player game each player plays a control of the form

\[
\begin{align*}
    dX^i_t &= \bar{\alpha}_t(X^i_t, m^N_{X_t}) dt + dB^i_t + \beta dW_t, \quad \text{with} \ X^i_0 \ \text{i.i.d. of law} \ \bar{m}_0.
\end{align*}
\]

(Without common noise \((\beta = 0)\)). Then \((m^N_{X_t})\) converges in law to a flow of measures \((m_t)\) given by \(m_t = \mathcal{L}(X_t)\) where \((X_t)\) solves

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The formal mean field limit game ($N \to +\infty$): the limit problem

- **Infinitely many small players,**
  (represented through their conditional distribution $m = (m_t)$ on $x$ given $W$ with initial distribution $\bar{m}_0$)

- **Dynamics of a representative player:** $dX_t = \alpha_t dt + dB_t + \beta dW_t$.

- **Goal of the small player:** Given $(m_t)$, to minimize over $\alpha$ the cost
  \[
  J(\alpha; (m_t)) = \mathbb{E} \left[ \int_0^T L(X_t, \alpha_t(X_t); m_t) dt + G(X_T, m_T) \right],
  \]

- **Nash equilibrium:** $\bar{\alpha}$ minimum of $\alpha \to J(\alpha; (\bar{m}_t))$ where
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  \bar{m}_t = \mathcal{L}(\bar{X}_t | W), \quad d\bar{X}_t = \bar{\alpha}_t(\bar{X}_t) dt + dB_t + \beta dW_t, \quad \mathcal{L}(\bar{X}_0) = \bar{m}_0.
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  \]
The PDE formulation for the MFG problem without common noise ($\beta = 0$)

- Let us introduce the value function of a typical player (given $(m_t)$):
  
  $$u(t_0, x_0) = \inf_{\alpha} \mathbb{E} \left[ \int_{t_0}^T L(X_t, \alpha_t(X_t), m_t) dt + G(X_T, m_T) \right],$$

  where $dX_t = \alpha_t dt + dB_t$, $X_{t_0} = x_0$.

- Then $u$ solves the Hamilton-Jacobi equation
  
  $$-\partial_t u - \frac{1}{2} \Delta u + H(x, Du, m_t) = 0, \quad u(T, x) = G(X_T, m_T),$$

  where $H(x, p, m) = \sup_{a} (-L(x, a, m) - p \cdot a)$

- And the optimal control is given by $\alpha^*_t(x) = -D_p H(x, Du, m_t)$.

- At the Nash equilibrium, $m_t = \mathcal{L}(X^*_t)$ where $dX_t = \alpha^*_t(X^*_t) dt + dB_t$, $X^*_0 \sim \bar{m}_0$ solves the Fokker-Planck equation
  
  $$\partial_t m - \frac{1}{2} \Delta m + \text{div}(\alpha^* m) = 0, \quad m_0 = \bar{m}_0.$$
The PDE formulation for the MFG problem without common noise \((\beta = 0)\)

Let us introduce the **value function** of a typical player (given \((m_t)\)):

\[
u(t_0, x_0) = \inf_{\alpha} \mathbb{E}\left[ \int_{t_0}^{T} \left( L(X_t, \alpha_t(X_t), m_t) \right) dt + G(X_T, m_T) \right],\]

where \(dX_t = \alpha_t dt + dB_t, \ X_{t_0} = x_0\).

Then \(u\) solves the **Hamilton-Jacobi equation**

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-\partial_t u - \frac{1}{2} \Delta u + H(x, Du, m_t) = 0, \quad u(T, x) = G(X_T, m_T),
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At the Nash equilibrium, \(m_t = \mathcal{L}(X^*_t)\) where \(dX_t = \alpha^*_t(X^*_t) dt + dB_t, X^*_0 \sim \bar{m}_0\) solves the Fokker-Planck equation

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\partial_t m - \frac{1}{2} \Delta m + \text{div}(\alpha^* m) = 0, \quad m_0 = \bar{m}_0
\]
Basic results of the MFG system \textbf{without common noise} \((\beta = 0)\)

For the \textbf{MFG equilibrium system}:

\[
\begin{cases}
  (i) & -\partial_t u - \Delta u + H(x, Du, m(t)) = 0 \quad \text{in } [0, T] \times \mathbb{R}^d \\
  (ii) & \partial_t m - \Delta m - \text{div}(mD_p H(x, Du, m)) = 0 \quad \text{in } [0, T] \times \mathbb{R}^d \\
  (iii) & m(0, \cdot) = \bar{m}_0, \quad u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{R}^d
\end{cases}
\]

- \textbf{Existence of solutions}: holds under general conditions (Lasry-Lions)
- \textbf{Uniqueness} cannot be expected in general, but holds
  - for small couplings or in a short time horizon (Huang-Caines-Malhamé, Lasry-Lions)
  - under a monotonicity conditions (Lasry-Lions): if \(H = H(x, p) - f(x, m)\) and
    \[
    \int_{\mathbb{R}^d} (f(x, m) - f(x, m'))d(m - m') \geq 0, \quad \int_{\mathbb{R}^d} (g(x, m) - g(x, m'))d(m - m') \geq 0.
    \]

- \textbf{Link with the \(N\)-player game}
  - from the MFG system to the \(N\)-player differential games
    Many contributions (Huang-Caines-Malhamé, Carmona-Delarue, ...)
  - from Nash equilibria of \(N\)-player differential games to the MFG system.
    - Open loop NE (Fischer, Lacker,...),
    - Closed loop NE (C.-Delarue-Lasry-Lions, Lacker, Djete,..).

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October 2021 9/22
Various aspects of MFG (selected topics)

- Other form of control problems
  - Mean field control, MFG of control
  - Optimal stopping (Bertucci,...)
  - Exit-time problems (Mazanti-Santambrogio,...)
  - MFG on networks (Camilli, Achdou-Dao-Ley-Tchou,...)
  - ...

- Variational formulation
  (Lasry-Lions, C.-Graber, Santambrogio et al.,...)

- Numerical aspects (Achdou-Capuzzo Dolcetta, Silva, Chassagneux-Crisan-Delarue, ...)

- Master equation, minor-major problems

- Learning (C.-Hadikhanloo, Elie and al., ...)
1. A short survey of MFG

2. MFG with common noise
Aim of this part

Our aim is to discuss Mean Field Game (MFG) systems with a common noise but no idiosyncratic noise (with unknown $(\bar{u}, \bar{v}, \bar{m})$):

\begin{align*}
d\bar{u}_t &= \{-\beta \Delta \bar{u}_t + H(D\bar{u}_t, x) - F(x, \bar{m}_t) - \sqrt{2\beta \text{div}(\bar{v}_t)}\}dt + \bar{v}_t \cdot \sqrt{2\beta} dW_t \quad \text{in } \mathbb{R}^d \times (0, T) \\
d\bar{m}_t &= \{\beta \Delta \bar{m}_t + \text{div}(\bar{m}_t D_p H(D\bar{u}_t, x))\}dt - \text{div}(\bar{m}_t \sqrt{2\beta} dW_t) \quad \text{in } \mathbb{R}^d \times (0, T) \\
\bar{m}_{t=0} &= \bar{m}_0, \quad \bar{u}_T(x) = G(x, \bar{m}_T) \quad \text{in } \mathbb{R}^d.
\end{align*}

- **Motivation:** Heterogeneous agent models, which often contain common noise terms but are seldom uniformly elliptic.

- **Difficulty:** no regularity of the solution, despite the Laplacian!
A MFG model with a common noise

- A typical player controls her dynamics

\[ dX_t^\alpha = \alpha_t dt + \sqrt{2\beta} dW_t \text{ in } [0, T] \quad X_0^\alpha = Z, \]

with \( \alpha \) an admissible control, \( W \) is the common noise, \( Z \) is the initial distribution. (Rk: no idiosyncratic noise)

- The cost of player a typical player is

\[ J(\alpha) = \mathbb{E} \left[ \int_0^T (L(\alpha_t, X_t^\alpha) + F(X_t^\alpha, m_t)) dt + g(X_T^\alpha, m_T) \right], \]

\( m_t \) is the conditional distribution of agents given the common noise \( W \)

- \( L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is convex in the first variable,
- \( F, G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \) are continuous.
The mean field equilibrium (probabilistic approach)

Definition

A pair \((\bar{\alpha}, \bar{m})\) is a (strong) MFG equilibrium if

1. \(\bar{\alpha}\) and \(\bar{m}\) are adapted to the filtration generated by \(Z\) and \(W\),

2. \(\bar{\alpha}\) is minimizes

\[
J(\alpha, \bar{m}) := \mathbb{E} \left[ \int_0^T \left( L(\alpha_t, X_t^{\alpha}) + F(X_t^{\alpha}, \bar{m}_t) \right) dt + g(X_T^{\alpha}, \bar{m}_T) \right],
\]

where \(dX_t^{\alpha} = \alpha_t dt + \sqrt{2\beta} dW_t, X_0^{\alpha} = Z\),

3. The compatibility condition holds:

\[
\bar{m}_t = \mathcal{L}(X_t^{\bar{\alpha}} | W) \quad \forall t \in [0, T].
\]

In a weak MFG equilibrium, \(\bar{\alpha}\) and \(\bar{m}\) are adapted to a larger filtration and \(\bar{m}_t\) is the conditional law of \(X_t^{\bar{\alpha}}\) given \(W\) and an additional noise.
The mean field equilibrium (PDE formulation)

Introducing the value function \((\bar{u}_t)\) of the problem, we see that the pair \((\bar{\alpha}, \bar{m})\) is a (strong) MFG equilibrium if

\[
\bar{\alpha}_t = -D_pH(D\bar{u}_t(x), x),
\]

where \((\bar{u}, \bar{v}, \bar{m})\) solves the stochastic MFG system

\[
\begin{align*}
    d\bar{u}_t &= \{-\beta \Delta \bar{u}_t + H(D\bar{u}_t, x) - F(x, \bar{m}_t) - 2\beta \text{div}(\bar{v}_t)\}dt + \bar{v}_t \cdot \sqrt{2\beta}dW_t & \text{in } \mathbb{R}^d \times (0, T) \\
    d\bar{m}_t &= \{\beta \Delta \bar{m}_t + \text{div}(\bar{m}_t D_p H(D\bar{u}_t, x))\}dt - \text{div}(\bar{m}_t \sqrt{2\beta}dW_t) & \text{in } \mathbb{R}^d \times (0, T) \\
    \bar{m}_{t=0} &= \bar{m}_0, \quad \bar{u}_T(x) = G(x, \bar{m}_T) & \text{in } \mathbb{R}^d.
\end{align*}
\]

where \(H(p, x) = \sup_\alpha -p \cdot \alpha - L(\alpha, x)\).

The vector field \(\bar{v}\) ensures the solution \(\bar{u}\) of the backward HJ equation to be adapted. (Peng ('92))
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- The vector field \(\bar{v}\) ensures the solution \(\bar{u}\) of the backward HJ equation to be adapted.
  (Peng (’92))
A few references

- **Statement of the problem**: Lasry & Lions ('05), Lions ('10), Bensoussan, Frehse & Yam ('13), Carmona & Delarue ('14).

- **Existence of a solution by a probabilistic approach**: Carmona, Delarue & Lacker ('16), . . ., Lacker ('21)
  - existence of a weak MFG equilibrium
  - existence and uniqueness of a strong MFG equilibrium (LL monotonicity condition + uniqueness of optimal controls)
  - weak MFG equilibria as limits of \(N\)-player games (with idiosyncratic noise).

- **Existence of a solution by a PDE approach**: C., Delarue, Lasry & Lions ('19), . . ., Gangbo, Mészáros, Mou & Zhang ('21)
  - classical solutions to the stochastic MFG system (monotony + idiosyncratic noise)
  - classical solutions to the master equation (monotony + idiosyncratic noise or convexity)

- **Goal of the talk**: analysis of the stochastic MFG system without idiosyncratic noise.
  - main issue: no smooth solutions
A change of variables

Let \((\bar{u}, \bar{v}, \bar{m})\) be a solution of the stochastic MFG system:

\[
\begin{align*}
    d\bar{u}_t &= \{-\beta \Delta \bar{u}_t + H(D\bar{u}_t, x) - F(x, \bar{m}_t) - 2\beta \text{div}(\bar{v}_t)\}dt + \bar{v}_t \cdot \sqrt{2\beta}dW_t & \text{in } \mathbb{R}^d \times (0, T) \\
    d\bar{m}_t &= \{\beta \Delta \bar{m}_t + \text{div}((\bar{m}_tD_pH(D\bar{u}_t, x)))\}dt - \text{div}(\bar{m}_t \sqrt{2\beta}dW_t) & \text{in } \mathbb{R}^d \times (0, T) \\
    \bar{m}_{t=0} &= \bar{m}_0, \quad \bar{u}_T(x) = G(x, \bar{m}_T) & \text{in } \mathbb{R}^d.
\end{align*}
\]

Setting

\[
\tilde{u}_t(x) = \bar{u}_t(x + \sqrt{2\beta}W_t, x) \text{ and } \tilde{m}_t = (id - \sqrt{2\beta}W_t)\circ \bar{m}_t,
\]

we obtain the new system (with unknown \((\tilde{u}, \tilde{M}, \tilde{m})\)):

\[
(s - \text{MFG}) \begin{cases}
    d\tilde{u}_t = \{\tilde{H}_t(D\tilde{u}_t(x), x) - \tilde{F}_t(x, \tilde{m}_t)\} dt + d\tilde{M}_t & \text{in } \mathbb{R}^d \times (0, T), \\
    \partial_t \tilde{m}_t = \text{div}(\tilde{m}_tD_p\tilde{H}(D\tilde{u}_t(x), x))dt & \text{in } \mathbb{R}^d \times (0, T), \\
    \tilde{m}_{t=0} = \bar{m}_0, \quad \tilde{u}_T = \tilde{G}(\cdot, \tilde{m}_T) & \text{in } \mathbb{R}^d
\end{cases}
\]

where \((\tilde{M}_t(x))_{t \in [0, T]}\) is an unknown martingale for a.e. \(x \in \mathbb{R}^d\) and where

\[
\begin{align*}
    \tilde{H}_t(x, p) &= H_t(p, x + \sqrt{2\beta}W_t), \\
    \tilde{F}_t(x, m) &= F(x + \sqrt{2\beta}W_t, (id + \sqrt{2\beta}W_t)\#m), \\
    \tilde{G}(x, m) &= G(x + \sqrt{2\beta}W_T, (id + \sqrt{2\beta}W_T)\#m).
\end{align*}
\]
The stochastic MFG system after the change of variables

\[
\begin{aligned}
\begin{cases}
d_t\tilde{u}_t = \left\{ \tilde{H}_t(D\tilde{u}_t(x), x) - \tilde{F}_t(x, \tilde{m}_t) \right\} dt + d\tilde{M}_t & \text{in } \mathbb{R}^d \times (0, T), \\
\partial_t\tilde{m}_t = \text{div}(\tilde{m}_t D_p \tilde{H}(D\tilde{u}_t(x), x)) dt & \text{in } \mathbb{R}^d \times (0, T), \\
\tilde{m}_0 = \bar{m}_0 & \tilde{u}_T = \tilde{G}(\cdot, \tilde{m}_T) & \text{in } \mathbb{R}^d
\end{cases}
\end{aligned}
\]

- The first equation is a Hamilton-Jacobi equation with random coefficients
  - Peng (’92): 2nd order backward HJ under a uniform ellipticity assumption,
  - Qiu (’18), Qiu and Wei (’19): viscosity solution involving derivatives on the path space.

  → Here one needs \( D\tilde{u} \) for the second equation.

- The second equation is a continuity equation with a nonsmooth drift
  - Di Perna and Lions (’89), Ambrosio (’04), Bouchut, James and Mancini (’05)...

Pierre Cardaliaguet

Some aspects of MFG

October 2021 18/22
The triplet $(\tilde{u}, \tilde{m}, \tilde{M})$ is a solution of (s-MFG) if:

1. $(\tilde{u}, \tilde{M}, \tilde{m})$ are adapted to $W$ and $(\tilde{M}_t(x))$ is a continuous martingale for a.e. $x \in \mathbb{R}^d$.
2. (regularity)\[ \|\tilde{m}\|_\infty + \|\tilde{u}_t\|_{W^{1,\infty}(\mathbb{R}^d)} + D^2\tilde{u}_t \cdot z \cdot z + \|\tilde{M}_t\|_\infty \leq C, \]
3. (Eq. for $\tilde{u}$) for a.e $(x, t)$ and $\mathbb{P}-a.s.$ in $\omega$,
   \[ \tilde{u}_t(x) = \tilde{G}(x, \tilde{m}_T) - \int_t^T (\tilde{H}_s(D\tilde{u}_s(x), x) - \tilde{F}_s(x, \tilde{m}_s))ds - \tilde{M}_T(x) + \tilde{M}_t(x), \]
4. (Eq. for $\tilde{m}$) in the sense of distributions and $\mathbb{P}-a.s.$ in $\omega$,
   \[ d_t\tilde{m}_t = \text{div}(\tilde{m}_t D_p\tilde{H}_t(D\tilde{u}_t, x)) \text{ in } \mathbb{R}^d \times (0, T) \quad \tilde{m}_0 = \bar{m}_0 \text{ in } \mathbb{R}^d. \]

**Theorem (C.-Souganidis)**

Under suitable assumptions (monotony of $\tilde{F}$ and $\tilde{G}$ and strict convexity of $\tilde{H}$), there exists a unique solution of (s-MFG).
The result relies on

- A new comparison result for HJ equation with random coefficients (Inspired by Douglis ('61))

- Discretization of the noise (cf. Carmona-Delarue-Lacker ('16)).

- Uniqueness of optimal solutions: if $\alpha^*_t(x) = -Dp\tilde{H}_t(D\tilde{u}_t(x), x)$, then the equation

$$dX_t = \alpha^*_t(X_t)dt + \sqrt{2\beta}dW_t, \quad X_0 = x_0$$

has a unique solution for a.e. $x_0$

Application to games with a finitely many of players.
The master equation

The master equation associated with our MFG with common noise is:

$$\begin{align*}
\partial_t U(t, x, m) & - \beta \Delta U(t, x, m) + H(D_x U(t, x, m), x) \\
+ \int_{\mathbb{R}^d} D_m U(t, x, m, y) \cdot D_p H(D_x U(t, y, m), y) m(dy) \\
- \beta \left( \int_{\mathbb{R}^d} \text{Tr}(D^2_{ym} U(t, x, m, y)) m(dy) + 2 \int_{\mathbb{R}^d} \text{Tr}(D^2_{xm} U(t, x, m, y)) m(dy) \right) \\
+ \int_{\mathbb{R}^{2d}} \text{Tr}(D^2_{mm} U(t, x, m, y, y') m(dy) m(dy')) & = F(x, m) \text{ in } \mathbb{R}^d \times \mathcal{P}_2.
\end{align*}$$

Theorem (C.-Souganidis)

Under suitable assumptions (monotony of $\tilde{F}$ and $\tilde{G}$ and strict convexity of $\tilde{H}$), there exists a unique weak solution to the master equation.

By “weak solution” we mean:

- After lifting to the space of probability measures (Lions)
- Weak formulation inspired by Bertucci ('20, '21)
Conclusion and open problems

In this talk,

- we discussed the backward (first order) stochastic HJ,
- we discussed the stochastic MFG system with common noise and no idiosyncratic noise,
- we applied the results to differential games with a large number of players,
- we discussed the existence/uniqueness of a solution to the associated Master equation.

Open questions:

- stochastic MFG with degenerate idiosyncratic noise and common noise (in progress with B. Seeger and P. Souganidis),
- identification of the martingale part,
- mean field limit of the Nash system with finitely many players.

Thank you!
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