Some aspects of MFG

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Mean Field Games (MFG) study collective behavior of rational agents.

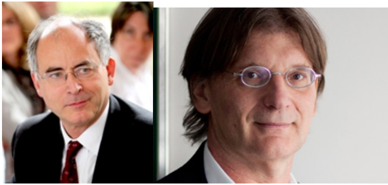
- collective behavior = infinitely many agents, having individually a negligible influence on the global system
- rational agents = each agent controls his state in order to minimize a cost which depends on the other agents' positions

Some references:

— Early work by Lasry-Lions (2006) and Caines-Huang-Malhamé (2006)

— Similar models in the economic literature: heterogeneous agent models (Aiyagari ('94), Krusell-Smith ('98),...)

— Recent monographs by Carmona-Delarue ('18) and Achdou-C.-Delarue-Porretta-Santambrogio ('20)



Outline

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1. A short survey of MFG

2. MFG with common noise

The $N-$player game

- N players
- Dynamics: $dX_t^i = \alpha_t^i dt + dB_t^i + \beta dW_t$,
  (where the B^i and W are i.i.d. B.M., α^i is the control of Player i, and X_0^i are i.i.d. of law $\bar{m}_0$)
- Goal of the players: **to minimize** over α^i the cost

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbb{E}\left[\int_0^T L(X_t^i, \alpha_t^i, m_{\mathbf{X}_t}^{N,i})dt + G(X_T^i, m_{\mathbf{X}_T}^{N,i})\right],$$

  where $m_{\mathbf{x}}^{N,i} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$ if $\mathbf{x} = (x_1, \dots, x_N)$.
- Nash equilibrium: $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ s.t., for any $i \in \{1, \dots, N\}$, $\bar{\alpha}^i$ minimizes $\alpha^i \to J^i(\alpha^i, (\bar{\alpha}^j)_{j\neq i})$.

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The formal mean field limit game ($N \to +\infty$): limit of dynamics

- Assume that in the $N-$player game each player plays a control of the form

$$dX_t^i = \bar{\alpha}_t(X_t^i, m_{\mathbf{X}_t}^N)dt + dB_t^i + \beta dW_t, \qquad \text{with } X_0^i \text{ i.i.d. of law } \bar{m}_0.$$

- (Without common noise ($\beta = 0$)). Then $(m_{\mathbf{X}_t}^N)$ converges in law to a flow of measures (m_t) given by $m_t = \mathcal{L}(X_t)$ where (X_t) solves

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- (With common noise ($\beta \neq 0$)). Then $(m_{\mathbf{X}_t}^N)$ converges in law to a flow of random measures (m_t) given by $m_t = \mathcal{L}(X_t \mid W)$ where (X_t) solves

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The formal mean field limit game ($N \to +\infty$): the limit problem

- Infinitely many small players,
  (represented through their conditional distribution $m = (m_t)$ on x given W with initial distribution $\bar{m}_0$)

- Dynamics of a representative player: $dX_t = \alpha_t dt + dB_t + \beta dW_t$.

- Goal of the small player: **Given** (m_t), to **minimize** over α the cost

$$J(\alpha; (m_t)) = \mathbb{E}\left[\int_0^T L(X_t, \alpha_t(X_t), m_t)dt + G(X_T, m_T)\right],$$

- Nash equilibrium: $\bar{\alpha}$ minimum of $\alpha \to J(\alpha; (\bar{m}_t))$ where

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The PDE formulation for the MFG problem without common noise ($\beta = 0$)

- Let us introduce the value function of a typical player (given (m_t)):

$$u(t_0, x_0) = \inf_{\alpha} \mathbb{E}\left[\int_{t_0}^{T} L(X_t, \alpha_t(X_t), m_t)dt + G(X_T, m_T)\right],$$

where $dX_t = \alpha_t dt + dB_t$, $X_{t_0} = x_0$,

- Then u solves the Hamilton-Jacobi equation

$$-\partial_t u - \frac{1}{2}\Delta u + H(x, Du, m_t) = 0, \qquad u(T, x) = G(X_T, m_T),$$

where $H(x, p, m) = \sup_a(-L(x, a, m) - p \cdot a)$

- And the optimal control is given by $\alpha_t^*(x) = -D_p H(x, Du, m_t)$.
- At the Nash equilibrium, $m_t = \mathcal{L}(X_t^*)$ where $dX_t = \alpha_t^*(X_t^*)dt + dB_t$, $X_0^* \sim \bar{m}_0$ solves the Fokker-Planck equation

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For the **MFG equilibrium system:**

$$(MFG)\quad \left\{ \begin{array}{lll} (i) & -\partial_t u - \Delta u + H(x, Du, m(t)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(mD_pH(x, Du, m)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ (iii) & m(0,\cdot) = \bar{m}_0,\ u(T,x) = g(x, m(T)) & \text{in } \mathbb{R}^d \end{array} \right.$$

- **Existence of solutions:** holds under general conditions (Lasry-Lions)
- **Uniqueness** cannot be expected in general, but holds
  - for small couplings or in a short time horizon (Huang-Caines-Malhamé, Lasry-Lions)
  - under a monotonicity conditions (Lasry-Lions): if $H = H(x,p) - f(x,m)$ and

$$\int_{\mathbb{R}^d} (f(x,m) - f(x,m'))d(m-m') \geq 0,\ \int_{\mathbb{R}^d} (g(x,m) - g(x,m'))d(m-m') \geq 0.$$

- **Link with the $N-$player game**
  - from the MFG system to the $N-$player differential games
    Many contributions (Huang-Caines-Malhamé, Carmona-Delarue, ...)
  - from Nash equilibria of $N-$player differential games to the MFG system.
    - Open loop NE (Fischer, Lacker,...),
    - Closed loop NE (C.-Delarue-Lasry-Lions, Lacker, Djete,..).

Various aspects of MFG (selected topics)

- Other form of control problems
  - Mean field control, MFG of control
  - Optimal stopping (Bertucci,...)
  - Exit-time problems (Mazanti-Santambrogio,...)
  - MFG on networks (Camilli, Achdou-Dao-Ley-Tchou,...)
  - ...
- Variational formulation
  (Lasry-Lions, C.-Graber, Santambrogio et al.,...)
- Numerical aspects (Achdou-Capuzzo Dolcetta, Silva, Chassagneux-Crisan-Delarue, ...)
- Master equation, minor-major problems
- Learning (C.-Hadikhanloo, Elie and al., ...)

Outline

Aim of this part

Our aim is to discuss Mean Field Game (MFG) systems with a common noise but no idiosyncratic noise (with unknown $(\bar{u}, \bar{v}, \bar{m})$):

$$\left\{\begin{array}{ll} d\bar{u}_t = \{-\beta\Delta\bar{u}_t + H(D\bar{u}_t, x) - F(x, \bar{m}_t) - \sqrt{2\beta}\mathrm{div}(\bar{v}_t)\}dt + \bar{v}_t \cdot \sqrt{2\beta}dW_t & \text{in } \mathbb{R}^d \times (0,T) \\ d\bar{m}_t = \{\beta\Delta\bar{m}_t + \mathrm{div}(\bar{m}_t D_pH(D\bar{u}_t, x))\}dt - \mathrm{div}(\bar{m}_t\sqrt{2\beta}dW_t) & \text{in } \mathbb{R}^d \times (0,T) \\ \bar{m}_{t=0} = \bar{m}_0, \qquad \bar{u}_T(x) = G(x, \bar{m}_T) & \text{in } \mathbb{R}^d. \end{array}\right.$$

- **Motivation:** Heterogeneous agent models, which often contain common noise terms but are seldom uniformly elliptic.
- **Difficulty:** no regularity of the solution, despite the Laplacian!

A MFG model with a common noise

- A typical player controls her dynamics

$$dX_t^\alpha = \alpha_t dt + \sqrt{2\beta} dW_t \text{ in } [0,T] \quad X_0^\alpha = Z,$$

with α an admissible control, W is the common noise, Z is the initial distribution. (Rk: no idiosyncratic noise)

- The cost of player a typical player is

$$J(\alpha) = \mathbb{E}\left[\int_0^T (L(\alpha_t, X_t^\alpha) + F(X_t^\alpha, m_t))dt + g(X_T^\alpha, m_T)\right],$$

  - (m_t) is the conditional distribution of agents given the common noise W
  - $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is convex in the first variable,
  - $F, G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are continuous.

The mean field equilibrium (probabilistic approach)

Definition

A pair $(\bar\alpha, \bar m)$ is a (strong) MFG equilibrium if

1. $\bar\alpha$ and $\bar m$ are adapted to the filtration generated by Z and W,
2. $\bar\alpha$ is minimizes

$$J(\alpha, \bar m) := \mathbb{E}\left[\int_0^T (L(\alpha_t, X_t^\alpha) + F(X_t^\alpha, \bar m_t)dt + g(X_T^\alpha, \bar m_T)\right],$$

   where $dX_t^\alpha = \alpha_t dt + \sqrt{2\beta} dW_t$, $X_0^\alpha = Z$,
3. The compatibility condition holds:

$$\bar m_t = \mathcal{L}(X_t^{\bar\alpha} \mid W) \qquad \forall t \in [0, T].$$

- In a weak MFG equilibrium, $\bar\alpha$ and $\bar m$ are adapted to a larger filtration and $\bar m_t$ is the conditional law of $X_t^{\bar\alpha}$ given W and an additional noise.

The mean field equilibrium (PDE formulation)

Introducing the value function $(\bar{u}_t)$ of the problem, we see that the pair $(\bar{\alpha}, \bar{m})$ is a (strong) MFG equilibrium if

$$\bar{\alpha}_t = -D_pH(D\bar{u}_t(x), x),$$

where $(\bar{u}, \bar{v}, \bar{m})$ solves the stochastic MFG system

$$\left\{ \begin{array}{l} d\bar{u}_t = \{-\beta\Delta\bar{u}_t + H(D\bar{u}_t, x) - F(x, \bar{m}_t) - 2\beta \mathrm{div}(\bar{v}_t)\}dt + \bar{v}_t \cdot \sqrt{2\beta}dW_t \qquad \text{in } \mathbb{R}^d \times (0, T) \\ d\bar{m}_t = \{\beta\Delta\bar{m}_t + \mathrm{div}(\bar{m}_t D_pH(D\bar{u}_t, x))\}dt - \mathrm{div}(\bar{m}_t\sqrt{2\beta}dW_t) \qquad \text{in } \mathbb{R}^d \times (0, T) \\ \bar{m}_{t=0} = \bar{m}_0, \qquad \bar{u}_T(x) = G(x, \bar{m}_T) \qquad \text{in } \mathbb{R}^d. \end{array} \right.$$

where $H(p, x) = \sup_\alpha -p \cdot \alpha - L(\alpha, x)$.

- The vector field $\bar{v}$ ensures the solution $\bar{u}$ of the backward HJ equation to be adapted. (Peng ('92))

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A few references

- Statement of the problem: Lasry & Lions ('05), Lions ('10), Bensoussan, Frehse & Yam ('13), Carmona & Delarue ('14).
- Existence of a solution by a probabilistic approach: Carmona, Delarue & Lacker ('16), . . . , Lacker ('21)
  $\longrightarrow$ existence of a weak MFG equilibrium
  $\longrightarrow$ existence and uniqueness of a strong MFG equilibrium (LL monotonicity condition+ uniqueness of optimal controls)
  $\longrightarrow$ weak MFG equilibria as limits of $N-$player games (with idiosyncratic noise).
- Existence of a solution by a PDE approach: C., Delarue, Lasry & Lions ('19), . . . , Gangbo, Mészáros, Mou & Zhang ('21)
  $\longrightarrow$ classical solutions to the stochastic MFG system (monotony+idiosyncratic noise)
  $\longrightarrow$ classical solutions to the master equation (monotony+idiosyncratic noise or convexity)
- **Goal of the talk:** analysis of the stochastic MFG system without idiosyncratic noise.
  $\longrightarrow$ main issue: no smooth solutions

A change of variables

Let $(\bar u, \bar v, \bar m)$ be a solution of the stochastic MFG system:

$$\begin{cases} d\bar u_t = \{-\beta\Delta \bar u_t + H(D\bar u_t, x) - F(x, \bar m_t) - 2\beta \mathrm{div}(\bar v_t)\}dt + \bar v_t \cdot \sqrt{2\beta}dW_t & \text{in } \mathbb{R}^d \times (0,T) \\ d\bar m_t = \{\beta\Delta \bar m_t + \mathrm{div}(\bar m_t D_pH(D\bar u_t, x))\}dt - \mathrm{div}(\bar m_t \sqrt{2\beta}dW_t) & \text{in } \mathbb{R}^d \times (0,T) \\ \bar m_{t=0} = \bar m_0, \qquad \bar u_T(x) = G(x, \bar m_T) & \text{in } \mathbb{R}^d. \end{cases}$$

Setting

$$\tilde u_t(x) = \bar u_t(x + \sqrt{2\beta}W_t, x) \;\text{ and }\; \tilde m_t = (id - \sqrt{2\beta}W_t)\sharp \bar m_t,$$

we obtain the new system (with unknown $(\tilde u, \tilde M, \tilde m)$):

$$(\mathbf{s}-\mathbf{MFG}) \begin{cases} d_t\tilde u_t = \left\{\tilde H_t(D\tilde u_t(x), x) - \tilde F_t(x, \tilde m_t)\right\} dt + d\tilde M_t \;\text{ in }\; \mathbb{R}^d \times (0,T), \\ \partial_t \tilde m_t = \mathrm{div}(\tilde m_t D_p \tilde H(D\tilde u_t(x), x))dt \;\text{ in }\; \mathbb{R}^d \times (0,T), \\ \tilde m_0 = \bar m_0 \qquad \tilde u_T = \tilde G(\cdot, \tilde m_T) \;\text{ in }\; \mathbb{R}^d \end{cases}$$

where $(\tilde M_t(x))_{t\in[0,T]}$ is an unknown martingale for a.e. $x \in \mathbb{R}^d$ and where

$$\begin{aligned} \tilde H_t(x,p) &= H_t(p, x + \sqrt{2\beta}W_t), \\ \tilde F_t(x,m) &= F(x + \sqrt{2\beta}W_t, (id + \sqrt{2\beta}W_t)\sharp m), \\ \tilde G(x,m) &= G(x + \sqrt{2\beta}W_T, (id + \sqrt{2\beta}W_T)\sharp m). \end{aligned}$$

The stochastic MFG system after the change of variables

$$
(\mathbf{s}-\mathbf{MFG}) \left\{ \begin{array}{l} d_t\tilde{u}_t = \left\{ \tilde{H}_t(D\tilde{u}_t(x), x) - \tilde{F}_t(x, \tilde{m}_t) \right\} dt + d\tilde{M}_t \ \text{ in } \ \mathbb{R}^d \times (0, T), \\ \partial_t \tilde{m}_t = \mathrm{div}(\tilde{m}_t D_p \tilde{H}(D\tilde{u}_t(x), x)) dt \ \text{ in } \ \mathbb{R}^d \times (0, T), \\ \tilde{m}_0 = \bar{m}_0 \qquad \tilde{u}_T = \tilde{G}(\cdot, \tilde{m}_T) \ \text{ in } \ \mathbb{R}^d \end{array} \right.
$$

- The first equation is a Hamilton-Jacobi equation with random coefficients
  - Peng ('92): 2nd order backward HJ under a uniform ellipticity assumption,
  - ...
  - Qiu ('18), Qiu and Wei ('19): viscosity solution involving derivatives on the path space.

  $\longrightarrow$ Here one needs $D\tilde{u}$ for the second equation.

- The second equation is a continuity equation with a nonsmooth drift
  - Di Perna and Lions ('89), Ambrosio ('04), Bouchut, James and Mancini ('05)...

The triplet $(\tilde{u}, \tilde{m}, \tilde{M})$ is a solution of **(s-MFG)** if:

1. $(\tilde{u}, \tilde{M}, \tilde{m})$ are adapted to W and $(\tilde{M}_t(x))$ is a continuous martingale for a.e. $x \in \mathbb{R}^d$,
2. (regularity)
$$\|\tilde{m}\|_\infty + \|\tilde{u}_t\|_{W^{1,\infty}(\mathbb{R}^d)} + D^2\tilde{u}_t\, z\cdot z + \|\tilde{M}_t\|_\infty \leq C,$$
3. (Eq. for $\tilde{u}$) for a.e (x,t) and $\mathbb{P}-$a.s. in ω,
$$\tilde{u}_t(x) = \tilde{G}(x, \tilde{m}_T) - \int_t^T (\tilde{H}_s(D\tilde{u}_s(x), x) - \tilde{F}_s(x, \tilde{m}_s))ds - \tilde{M}_T(x) + \tilde{M}_t(x),$$
4. (Eq. for $\tilde{m}$) in the sense of distributions and $\mathbb{P}-$a.s. in ω,
$$d_t\tilde{m}_t = \mathrm{div}(\tilde{m}_t D_p\tilde{H}_t(D\tilde{u}_t, x)) \;\text{ in }\; \mathbb{R}^d \times (0,T) \quad \tilde{m}_0 = \bar{m}_0 \;\text{ in }\; \mathbb{R}^d.$$

Theorem (C.-Souganidis)

Under suitable assumptions (monotony of $\tilde{F}$ and $\tilde{G}$ and strict convexity of $\tilde{H}$), there exists a unique solution of **(s-MFG)**.

Comments and applications

- The result relies on
  - A new comparison result for HJ equation with random coefficients (Inspired by Douglis ('61))
  - Discretization of the noise (cf. Carmona-Delarue-Lacker ('16)).
  - Uniqueness of optimal solutions: if $\alpha_t^*(x) = -D_p\tilde{H}_t(D\tilde{u}_t(x), x)$, then the equation

$$dX_t = \alpha_t^*(X_t)dt + \sqrt{2\beta}dW_t, \qquad X_0 = x_0$$

    has a unique solution for a.e. x_0

- Application to games with a finitely many of players.

The master equation

The master equation associated with our MFG with common noise is:

$$
\begin{aligned}
&\partial_t U(t,x,m) - \beta \Delta U(t,x,m) + H(D_x U(t,x,m), x) \\
&+ \int_{\mathbb{R}^d} D_m U(t,x,m,y) \cdot D_p H(D_x U(t,y,m), y) m(dy) \\
&- \beta \Big(\int_{\mathbb{R}^d} Tr(D^2_{ym} U(t,x,m,y)) m(dy) + 2 \int_{\mathbb{R}^d} Tr(D^2_{xm} U(t,x,m,y)) m(dy) \\
&+ \int_{\mathbb{R}^{2d}} Tr(D^2_{mm} U(t,x,m,y,y')) m(dy) m(dy') \Big) = F(x,m) \text{ in } \mathbb{R}^d \times \mathcal{P}_2.
\end{aligned}
$$

Theorem (C.-Souganidis)

Under suitable assumptions (monotony of $\tilde{F}$ and $\tilde{G}$ and strict convexity of $\tilde{H}$), there exists a unique weak solution to the master equation.

By "weak solution" we mean:

- After lifting to the space of probability measures (Lions)
- Weak formulation inspired by Bertucci ('20, '21)

Conclusion and open problems

In this talk,

- we discussed the backward (first order) stochastic HJ ,
- we discussed the stochastic MFG system with common noise and no idiosyncratic noise,
- we applied the results to differential games with a large number of players,
- we discussed the existence/uniqueness of a solution to the associated Master equation.

Open questions:

- stochastic MFG with degenerate idiosyncratic noise and common noise (in progress with B. Seeger and P. Souganidis),
- identification of the martingale part,
- mean field limit of the Nash system with finitely many players.

Thank you!

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